Topology Meets Number Theory

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by

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Reporting on joint research
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I wish to begin by thanking the organizers for inviting me to present at this workshop. It is my pleasure to be here, if only on zoom.

My first time in South Africa was 50 years ago, for my honeymoon. And I am pleased to say that my wife and I are still together, and now have three grandchildren.

I was honoured to serve as Acting President of the University of South Australia when my university conferred an Honorary Doctorate on Nelson Mandela.

The research I shall speak on today is with an early career researcher who obtained his PhD in South Africa, was a teaching assistant at AIMS, and will take up a university position in South Africa in 2024.

Most of the results in this presentation appear in the articles:

- Taboka Prince Chalebgwa and Sidney A. Morris, “Topology Meets Number Theory”, American Mathematical Monthly, (2024).
- Taboka Prince Chalebgwa and Sidney A. Morris, Sin, cos, exp, and log of Liouville numbers, Bulletin of the Australian Mathematical Society, (2023), 108 (1), 81–85. (Open access)
- Taboka Prince Chalebgwa and Sidney A. Morris, Erdős properties of the Mahler set S, Bull. Austral. Math. Soc. (2023), 108(3), 504–510. (Open access)

Subsets of the set $\mathbb{R}$ of all real numbers

- G_δ-*set*: countable intersection of open sets;
- F_σ-*set*: countable union of closed sets;
- *Borel set*: can be constructed from open sets using countable intersections, countable unions, and relative complements
  ($B \setminus A$ is the relative complement of A in B);
- *analytic set*: continuous image of a Borel set.

$$
\begin{tikzcd}
& \boxed{2^{\mathfrak{c}} \text{ subsets of } \mathbb{R}} & \\
& \boxed{2^{\mathfrak{c}} \text{ Lebesgue measurable subsets of } \mathbb{R}} \arrow[u] & \\
& \boxed{\mathfrak{c} \text{ analytic sets}} \arrow[u] & \\
& \boxed{\mathfrak{c} \text{ Borel sets}} \arrow[u] & \\
\boxed{\mathfrak{c}\ G_\delta\text{-sets}} \arrow[ur] & & \boxed{\mathfrak{c}\ F_\sigma\text{-sets}} \arrow[ul] \\
& \boxed{\mathfrak{c} \text{ open sets } \& \ \mathfrak{c} \text{ closed sets}} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

In the time available, I plan to discuss 8 topological & metric space and 10 number theory "concepts".

First the topological concepts we shall mention:

(1) homeomorphism;

(2) dense set;

(3) G_δ-space;

(4) Borel set;

(5) analytic space;

(6) Hausdorff dimension (a metric space concept);

(7) Cantor space, $\mathbb{G}$ (named after **G**eorg Cantor (1845–1918))

(8) topological space $\mathbb{P}$ of real irrational numbers.

From Number Theory:

(1) the set $\mathbb{P}$ of all real irrational numbers;

(2) the set $\mathcal{T}$ of all real transcendental numbers;

(3) the set $\mathbb{Q}$ of rational numbers;

(4) the set $\mathbb{A}$ of real algebraic numbers;

(5) the set $\mathbb{R}$ of all real numbers;

(6) algebraically independence;

(7) transcendence basis;

(8) irrationality exponent;

(9) Liouville numbers $\mathcal{L}$;

(10) Mahler sets $\mathbb{A}$, S, T, U.

In 1844 Joseph Liouville (1809–1882) was the first to prove that transcendental numbers exist. Indeed, he introduced an uncountable set $\mathcal{L}$ of real transcendental numbers, known as *Liouville numbers*.

The first example was the **Liouville constant** $\ell = \sum_{n=1}^{\infty} 10^{-n!}$; that is the real number with the digit in the nth decimal place equal to 0, unless $n = k!$, $k = 1, 2, \ldots,$ in which case it equals 1.

We note that $\mathcal{L}$ is dense in $\mathbb{R}$, is uncountable but has Lebesgue measure equal to zero and is totally disconnected. (In fact it has Hausdorff dimension equal to zero.) So the set $\mathcal{L}$ is some sense "small".

In his book "Joseph Liouville 1809–1882, Master of Pure and Applied Mathematics" Jersper Lützen claims Liouville was the most important French mathematician in the generation between Évariste Galois (1811–1832) and Charles Hermite (1822–1901).

Definition 1. A real number ξ is called a **Liouville number** if for every positive integer n, there exists a pair of integers (p, q) with $q > 1$, such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^n}.$$

We see that the Liouville constant ℓ is a Liouville number.

Our first theorem, proved by Paul Erdős (1913–1996) in 1962, is surprising in that $\mathcal{L}$ is a small set

This photo was taken by Sid Morris in 1979 when he was visiting the University of Calgary. Paul Erdős was visiting his friend and colleague Richard Guy.

Definition 2. Let X be a subset of $\mathbb{R}$. The set X is said to have the **Erdős property** if for each $r \in \mathbb{R}$ there exist $x_1, x_2 \in X$ such that $r = x_1 + x_2$. The set X is said to have the **multiplicative Erdős property** if for every $s \in \mathbb{R}, s > 0$ there exist $x_3, x_4 \in X$ such that $s = x_3 \cdot x_4$.

Theorem 1. The set $\boldsymbol{\mathcal{L}}$ has the Erdős property and the multiplicative Erdős property. Indeed if $\boldsymbol{X}$ is any dense $\boldsymbol{G_\delta}$-subset of $\mathbb{R}$ then it has the Erdős property and the multiplicative Erdős property.

It follows from the Baire Category Theorem that the intersection of any two (or even a countable number of) dense G_δ-subsets of $\mathbb{R}$ is a G_δ-subset of $\mathbb{R}$.

This result immediately yields Theorem 2.

Theorem 2. [CM] Every dense $\boldsymbol{G_\delta}$-subset of $\mathbb{R}$ contains an uncountable subset of $\boldsymbol{\mathcal{L}}$.

A purely topological property of a set $\Longrightarrow$ it contains an uncountable number of transcendental numbers.

We now recall a beautiful characterization of the space $\mathbb{P}$ of all irrational real13 numbers. This result can be found in the 2001 book "The Infinite-Dimensional Topology of Function Spaces" by John van Mill.

Theorem. The space $\mathbb{P}$ of all irrational real numbers is topologically the unique nonempty, separable, metrisable, topologically complete, nowhere locally compact, and zero-dimensional space (where a topological space is said to be *nowhere locally compact* if no point of it has a neighbourhood with compact closure.)

Theorem 3. Every dense G_δ-subset of $\mathbb{R}$ is homeomorphic to $\mathbb{P}$ and to $\mathbb{N}^{\aleph_0}$.
In particular, this is the case for the set $\mathcal{T}$ of all real transcendental numbers and for $\mathcal{L}$.

Theorem 3. Every dense G_δ-subset of $\mathbb{R}$ is homeomorphic to $\mathbb{P}$. In particular, this is the case for the set $\mathcal{T}$ of all real transcendental numbers and for $\mathcal{L}$.

Observe that set $\mathbb{P}$ contains the set $\mathcal{L}$ and the cardinality of $\mathbb{P} \setminus \mathcal{L}$ is $\mathfrak{c}$. This immediately gives us:

Theorem 4. [CM] Every dense G_δ-subset X of $\mathbb{R}$ contains a dense G_δ-subset Y of $\mathbb{R}$ such that the set $X \setminus Y$ has cardinality $\mathfrak{c}$.

This answers a question of Erdős.

Erdős searched for a proper subset of $\mathcal{L}$ which has the Erdős property. From Theorem 4 we know that $\mathcal{L}$ contains a chain $L_1, L_2, \ldots, L_n, \ldots$ such that

$$\mathcal{L} \supset L_1 \supset L_2 \supset \cdots \supset L_n \supset \ldots$$

with each L_n being a dense G_δ-subset of $\mathbb{R}$ and so having the Erdős property. So there is no smallest set with the Erdős property. Indeed as $\mathcal{L} \setminus L_1$ has cardinality $\mathfrak{c}$, if Y is any of the $2^{\mathfrak{c}}$ subsets of $\mathcal{L} \setminus L_1$, then $L_1 \cup Y$ has the Erdős property.

Theorem 5. [CM] There exist $\mathbf{2^{\mathfrak{c}}}$-subsets of $\boldsymbol{\mathcal{L}}$ with the Erdős property.

The Gelfond-Schneider Theorem (1934) says: *if a and b are (complex) algebraic numbers with $a \neq 0, 1$ and b not a rational number, then a^b is a transcendental number.* In 2023 Diego Marques and Marcelo Oliveira extended this to when b is a Liouville number. By contrast:

Theorem 6. [CM] If $\boldsymbol{s}$ is any positive real number with $\boldsymbol{s \neq 1}$, then there exist $\boldsymbol{a, b \in \mathcal{L}}$, with $\boldsymbol{a, b > 0}$, such that $\boldsymbol{s = a^b}$. Indeed, if $\boldsymbol{X}$ is any dense $\boldsymbol{G_\delta}$-subset of $\mathbb{R}$, then $\boldsymbol{s = {x_1}^{x_2}}$, for some $\boldsymbol{x_1, x_2 \in X}$. Further, $\boldsymbol{X}$ can be replaced by any subset of $\mathbb{R}$ of full measure, such as the set of normal numbers.

Theorem 6A. [CM] If $\boldsymbol{s}$ is any positive real number with $\boldsymbol{s \neq 1}$, then there exist $\boldsymbol{a, b \in \mathcal{L}}$, with $\boldsymbol{a, b > 0}$, such that $\boldsymbol{s = a^b}$.

Proof.

If s is any positive real number such that $s \neq 1$, put $r = \frac{1}{\log_e(s)}$. So $s = \exp \frac{1}{r}$. Then $f(x) = r \log_e(x)$ is a homeomorphism of $(0, \infty)$ onto $(-\infty, \infty)$.

Proof. If s is any positive real number such that $s \neq 1$, put $r = \frac{1}{\log_e(s)}$. So $s = \exp \frac{1}{r}$. Then $f(x) = r \log_e(x)$ is a homeomorphism of $(0, \infty)$ onto $(-\infty, \infty)$.

The set $\mathcal{L}^+$ of positive numbers in $\mathcal{L}$ is $\mathcal{L} \cap (0, \infty)$ and is a dense G_δ-subset of $(0, \infty)$. Then $f(\mathcal{L}^+)$ is a dense G_δ-subset of the set of all real numbers, and so $f(\mathcal{L}^+)$ contains a dense G_δ subset L_0 of the set of all Liouville numbers.

So for any $l_2 \in L_0$, there is an $l_1 \in \mathcal{L}^+$ such that $f(l_1) = r \log_e(l_1) = l_2$.

Proof. If s is any positive real number such that $s \neq 1$, put $r = \frac{1}{\log_e(s)}$. So $s = \exp\frac{1}{r}$. Then $f(x) = r\log_e(x)$ is a homeomorphism of $(0, \infty)$ onto $(-\infty, \infty)$.

The set $\mathcal{L}^+$ of positive numbers in $\mathcal{L}$ is $\mathcal{L} \cap (0, \infty)$ and is a dense G_δ-subset of $(0, \infty)$. Then $f(\mathcal{L}^+)$ is a dense G_δ-subset of the set of all real numbers, and so $f(\mathcal{L}^+)$ contains a dense G_δ subset L_0 of the set of all Liouville numbers. So for any $l_2 \in L_0$, there is an $l_1 \in \mathcal{L}^+$ such that $f(l_1) = r\log_e(l_1) = l_2$.

As l_2 is a Liouville number, so too is $b = \frac{1}{l_2}$.

Thus $\frac{1}{r} = \log_e(a^b)$, where $a = l_1$.
Hence $\exp(\frac{1}{r}) = a^b = s$, as required. ■

It is clear from the proof of the previous theorem, that for each positive real number $s \neq 1$, there is an uncountable number of different pairs (a, b) which satisfy the theorem, since l_2 can be chosen to be any member of the set L_0, which is a dense G_δ-subset of $\mathcal{L}$ and hence also of $\mathbb{R}$, and so is uncountable.

In his influential book "Transcendental Number Theory" Fields Medalist Alan Baker (1939–2018) introduces the chapter on Mahler's Classification as follows: "A classification of the set of transcendental numbers into three distinct aggregates, termed S-, T-, and U-numbers, was introduced by Mahler in 1932, and it has proved to be of considerable value in the general development of the subject."

Kurt Mahler (1903–1988) was initially interested in proving that π and e were not Liouville numbers. He partitioned the complex numbers into four classes S-numbers, T-numbers, U-numbers and complex $\mathbb{A}$-numbers. The set of all Liouville numbers is a proper subset of U. Then it was shown that e is an S-number and π is either an S-number or a T-number. So neither e nor π is a Liouville number.

(It is still not known whether π is a T-number. If it is, then $e+\pi$ is transcendental. However, today it is not known whether it is even irrational.)

Given a polynomial $P(X) \in \mathbb{C}[X]$, the height of P, denoted by $H(P)$, is the maximum of the absolute values of the coefficients of P.
Given a complex number ξ, a positive integer n, and a real number $H \geq 1$, we define the quantity

$$w_n(\xi, H) = \min\{|P(\xi)| : P(X) \in \mathbb{Z}[X],\ H(P) \leq H,\ \deg(P) \leq n,\ P(\xi) \neq 0\}.$$

We set $\quad w_n(\xi) = \limsup_{H \to \infty} \frac{-\log w_n(\xi, H)}{\log H} \quad$ and

$$w(\xi) = \limsup_{n \to \infty} \frac{w_n(\xi)}{n}.$$

Do $w_n(\xi)$ & $w(\xi) = 0$, ∞, or something in-between?

Focussing on the set $\mathbb{R}$ of real numbers, Kurt Mahler partitions the set as follows:

Definition 3. Let ξ be a real number. The number ξ is

(i) an **$\mathbb{A}$-number** if $w(\xi) = 0$,

(ii) an **S-number** if $0 < w(\xi) < \infty$,

(iii) a **T-number** if $w(\xi) = \infty$ and $w_n(\xi) < \infty$ for any $n \geq 1$,

(iv) a **U-number** if $w(\xi) = \infty$ and $w_n(\xi) = \infty$ for all $n \geq n_0$, for some positive integer n_0.

The $\mathbb{A}$-numbers are the algebraic numbers and there exist an infinity of $\mathbb{A}$-numbers, S-numbers, U-numbers and T-numbers.

The set $\mathcal{L}$ of Liouville numbers is a proper subset of the set of U-numbers.

It was an open question for 36 years on whether the set of T-numbers is non-empty. It was answered in 1970 in the positive by Wolfgang M. Schmidt who won the Frank Nelson Cole Prize in Number Theory for work on Diophantine Approximation.

The set of S-numbers has full Lebesgue measure, while the set of U-numbers, T-numbers, and $\mathbb{A}$-numbers each have zero Lebesgue measure.

Definition 4. A subset S of a field $\mathbb{K}$ is said to be **algebraically independent over $\mathbb{K}$** if the elements of S do not satisfy any non-trivial polynomial equation with coefficients in $\mathbb{K}$.

The following theorem of Mahler records a fundamental property of the Mahler classes.

Theorem 7. If $\xi, \eta \in \mathbb{R}$ are algebraically dependent, then they belong to the same Mahler class.

The next beautiful theorem was proved using Mahler classes and the previous theorem.

Theorem 8. [CM] For any (real or complex) $\boldsymbol{U}$-number $\boldsymbol{\alpha}$, in particular for $\boldsymbol{\alpha}$ any Liouville number, all of the following are transcendental numbers:
$e^{\boldsymbol{\alpha}}$, $\mathbf{log}_e\, \boldsymbol{\alpha}$, $\mathbf{sin}\, \boldsymbol{\alpha}$, $\mathbf{cos}\, \boldsymbol{\alpha}$, $\mathbf{tan}\, \boldsymbol{\alpha}$, $\mathbf{sinh}\, \boldsymbol{\alpha}$, $\mathbf{cosh}\, \boldsymbol{\alpha}$, $\mathbf{tanh}\, \boldsymbol{\alpha}$
and the inverse functions evaluated at $\boldsymbol{\alpha}$ of the listed trigonometric and hyperbolic functions, noting that wherever multiple values are involved, each such value is transcendental.

In 1919 *Felix Hausdorff* (1868–1942) introduced the notion of **Hausdorff dimension of a metric space**. A surprising feature of Hausdorff dimension is that it can have values which are not integers. This topic was developed by *Abram Samoilovitch Besicovitch* (1891–1970) a decade or so later, but came into prominence in the 1970s with the work of*Benoit Mandelbrot* (1924–2010) on what he called fractal geometry and which spurred the development of chaos theory. Fractals and chaos theory have been used in a very wide range of disciplines including economics, finance, meteorology, physics, and physiology.

Hausdorff, Besicovitch, Mandelbrot

Definition 5. Let X be a metric space, $S \subset X$ and $d \in [0, \infty)$.

$$H_\delta^d(S) = \text{inf} \left\{ \sum_{i=1}^{\infty} (\text{diam } U_i)^d : \bigcup_{i=1}^{\infty} \supseteq S, \text{diam}\, U_i < \delta \right\}$$

where the infimum is taken over all countable covers U of S.

The restriction of $H^d(S) = \lim_{\delta \to 0} H_\delta^d(S)$ to measurable sets is said to be the **d-dimensional Hausdorff measure**.

The **Hausdorff dimension** is defined to be $\text{inf}\{d \geq 0 : H^d(X) = 0\}$.

Theorem 9. [Ki, H. (1995), Proc. Amer. Math. Soc.] Each of the Mahler sets is a Borel set.

The following powerful theorem combines the main result in 2002 of the paper "Hausdorff dimension, analytic sets and transcendence" by Gerald A Edgar and C. Miller in Real Anal. Exch. and a standard result from topology.

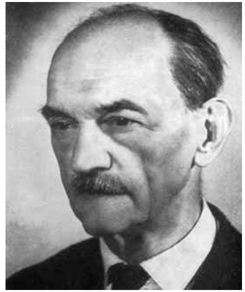
Theorem 10. If X is an uncountable analytic subset of $\mathbb{R}$, then it has a subspace homeomorphic to $\mathbb{G}$. In particular, X has cardinality $\mathfrak{c}$.
If Y is an analytic subset of $\mathbb{R}$ with finite positive Hausdorff dimension, then it has cardinality $\mathfrak{c}$ and contains a maximal algebraically independent subset of $\mathbb{R}$ (that is a transcendence basis for $\mathbb{R}$).

Definition 5. Let ξ be a real number. Then ξ is said to have **irrationality exponent** $m(\xi)$ if $m(\xi)$ is the infimum of the set R of all m such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^m}$$

has at most finitely-many solutions $\frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. If $R = \emptyset$, then ξ is said to have **infinite irrationality exponent** (which happens only for Liouville numbers).

It is routine to prove that each set of real numbers of irrationality exponent $m \in (2, \infty)$ is analytic.



The following theorem of Vojtēch Jarník (1897–1970) dates back almost 100 years, to 1929.

Theorem 11. The set of real numbers of irrationality exponent equal to 2 has full Lebesgue measure. The set of real numbers of irrationality exponent $m \in (2, \infty)$ has Lebesgue measure 0 and Hausdorff dimension equal to $\frac{2}{m}$.

Mahler had expressed an interest in the intersection of the middle-third Cantor set $\mathbb{G}$ with other sets. There is also a significant amount of literature on the intersection of the middle-third Cantor set with translations of it because, since Jules Henri Poincaré (1854–1912) in the late 1800s, it plays a role in studying nonlinear dynamical systems. We touch upon a couple of results on intersections with the middle-third Cantor set.

Theorem 12. [CM] For each $\boldsymbol{m} \in [\mathbf{2}, \infty)$, let $\boldsymbol{E_m}$ be the set of real numbers of irrationality exponent equal to $\boldsymbol{m}$. Then $\mathbb{G} \cap \boldsymbol{E_m}$ has cardinality $\mathfrak{c}$ and has subspace homeomorphic to $\mathbb{G}$.
For the Mahler set $\boldsymbol{S}$, $\mathbb{G} \cap \boldsymbol{S}$, is (infinite and) a dense subset of $\mathbb{G}$.

Theorem 13. [CM] The Cantor-Liouville set $\mathbb{G} \cap \boldsymbol{\mathcal{L}}$ has cardinality $\mathfrak{c}$. Further, $\mathbb{G} \cap \boldsymbol{\mathcal{L}}$ has a subspace homeomorphic to $\mathbb{G}$.
Indeed for any $\boldsymbol{q} \in \mathbb{Q}$, the set $(\boldsymbol{q} + \mathbb{G}) \bigcap \boldsymbol{\mathcal{L}}$ has cardinality $\mathfrak{c}$ and has a subspace homeomorphic to $\mathbb{G}$.

Finally, observing that we have shown that there exist $2^{\mathfrak{c}}$ subsets of $\mathcal{L}$ which have Lebesgue measure zero and have the Erdős property, we complement this with the following theorem:

Theorem 14. Let $\boldsymbol{m} \in [\mathbf{0}, \infty)$. Then there exist $\mathbf{2}^{\mathfrak{c}}$ dense subsets $\boldsymbol{W}$ of $\boldsymbol{S}$ each of Lebesgue measure $\boldsymbol{m}$ such that $\boldsymbol{W}$ has the Erdős property and no two of these $\boldsymbol{W}$ are homeomorphic. There also exist $\mathbf{2}^{\mathfrak{c}}$ dense subsets of $\boldsymbol{S}$ which have full Lebesgue measure, and no two of these are homeomorphic.

We note that Theorem 14 remains true if we consider the class of complex S-numbers and replace the Erdős property by the the **complex Erdős property** that every complex number is the sum of two numbers from the set W.

We leave as an **Open Question** whether Theorem 14 is true if the Mahler class S is replaced by the Mahler class T.

Dankie