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Sophus Lie's third fundamental theorem and the adjoint functor theorem

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The essential attributes of a Lie group G are the associated Lie algebra $\mathfrak{L}(G)$ and the exponential function $\exp : \mathfrak{L}(G) \to G$. The prescription $\mathfrak{L}$ operates not only on Lie groups but also on morphisms between them: it is a functor.

Many features of Lie theory are shared by classes of topological groups which are much larger than that of Lie groups; these classes include the classes of compact groups, locally compact groups, and pro-Lie groups, that is, complete topological groups having arbitrarily small normal subgroups N such that G/N is a (finite-dimensional) Lie group.

Considering the functor $\mathfrak{L}$ it is therefore appropriate to contemplate more general classes of topological groups. Certain functorial properties of the assignment of a Lie algebra to a topological group (where possible) will be essential. What is new here is that we will introduce a functorial assignment from Lie algebras to groups and investigate to what extent it is inverse to the Lie algebra functor $\mathfrak{L}$. While the Lie algebra functor is well known and is cited regularly, the existence of a Lie group functor available to be cited and applied appears less well known. Sophus Lie's Third Fundamental Theorem says that for each finite-dimensional real Lie algebra there is a Lie group whose Lie algebra is (isomorphic to) the given one; but even in classical circumstances it is not commonly known that this happens in a functorial fashion and what the precise relationship between the Lie algebra functor and the Lie group functor is.

1 The exponential function of topological groups

We shall utilize some basic category theory such as concerns the nature and existence of adjoint functors, and it helps if the reader knows the rudiments of Lie group theory such as Lie algebras and the exponential function.

Definitions 1.1. Let G be a topological group. A *one parameter subgroup* is a morphism $X : \mathbb{R} \to G$ of topological groups. The set $\mathrm{Hom}(\mathbb{R}, G)$ of all one parameter subgroups is given the topology of uniform convergence on compact sets, that is, basic neighborhoods of a one parameter subgroup $X : \mathbb{R} \to G$ are of the form $W(X; n, U)$, where $n \in \mathbb{N}$ and U ranges through the open identity neighborhoods of G and where

$$W(X; n, U) \stackrel{\text{def}}{=} \{Y \in \operatorname{Hom}(\mathbb{R}, G) \mid (\forall r \in \mathbb{R}, |r| \leqslant n)\, Y(r)X(r)^{-1} \in U\}.$$

The set $\operatorname{Hom}(\mathbb{R}, G)$ endowed with this topology is denoted by $\mathfrak{L}(G)$. The continuous evaluation function $X \mapsto X(1) : \mathfrak{L}(G) \to G$ will be denoted by $\exp_G$ and will be called the *exponential function* of the topological group G. We note that $\exp_G X = X(1)$.

A few immediate observations are in order. Firstly, let us say that an action $(r, x) \mapsto r \cdot x : \mathbb{R} \times X \to X$ of $\mathbb{R}$ on a Hausdorff topological space X with base-point x_0 is a *scalar multiplication* if the following conditions are satisfied:

(i) the action is continuous;

(ii) $(\forall x \in X)\ 0 \cdot x = x_0$ and $(\forall x \in X)\ 1 \cdot x = x$;

(iii) $(\forall r, s \in \mathbb{R}, x \in X)\ (rs) \cdot x = r \cdot (s \cdot x)$;

(iv) for each $x \in X$, the orbit $\mathbb{R} \cdot x$ is an abelian group with respect to an operation $+$ which satisfies $(\forall r, s \in \mathbb{R})\ (r \cdot x) + (s \cdot x) = (r + s) \cdot x$.

Let X be a Hausdorff topological space X with scalar multiplication and with base-point x_0. It is easily seen that if $x_0 \neq x \in X$ then the function $r \mapsto r \cdot x : \mathbb{R} \to \mathbb{R} \cdot x$ is a bijective morphism of abelian groups, that the base-point x_0 is the neutral element of all abelian groups $\mathbb{R} \cdot x$, $x \in X$ and that the multiplicative group $\mathbb{R}^\times = (\mathbb{R} \backslash \{0\}, \cdot)$ acts on X and has the orbits $\{x_0\}$ and $\mathbb{R}^\times \cdot x$ with $x \neq x_0$. Thus a topological space with a scalar multiplication looks a lot like a topological vector space without addition.

If $\mathbb{R}^n$ is a euclidean space and S any closed non-empty subset of the unit sphere, then $\mathbb{R} \cdot S$ is a locally compact space with a scalar multiplication (namely, the one induced by the ordinary scalar multiplication of $\mathbb{R}^n$).

The following remark will show that the topological space $\mathfrak{L}(G)$ has a scalar multiplication for every topological group G.

Remark 1.2. (i) The constant morphism $O : \mathbb{R} \to G$, $O(r) = 1$, is a member of $\mathfrak{L}(G)$ which we consider distinguished; so $\mathfrak{L}(G)$ is a pointed space, i.e. a space with a base-point.

(ii) Define a continuous action $\mathbb{R} \times \mathfrak{L}(G) \to \mathfrak{L}(G)$ by $(r \cdot X)(t) = X(tr)$ for $r, t \in \mathbb{R}$, $X \in \mathfrak{L}(G)$. Then $0 \cdot X = O$ and $(rs) \cdot X = r \cdot (s \cdot X)$.

(iii) For each $X \in \mathfrak{L}(G)$, the image $A \stackrel{\text{def}}{=} X(\mathbb{R})$ is an abelian subgroup of G, and thus $\mathfrak{L}(A) = \operatorname{Hom}(\mathbb{R}, A)$ is an abelian group under pointwise multiplication. Each $r \cdot X$ may be considered as an element of $\mathfrak{L}(A)$ such that $(r + s) \cdot X = (r \cdot X)(s \cdot X)$. The last identity can be rephrased as

$$\exp(r + s) \cdot X = (\exp r \cdot X)(\exp s \cdot X).$$

Proof. Almost all of these assertions are proved straightforwardly, but the continuity of the action deserves comment. Let $W(r_0 \cdot X_0; n, U)$ be a basic neighborhood of $r_0 \cdot X_0$ according to Definition 1.1. Then we have $Y \in W(r_0 \cdot X_0; n, V)$ if and only if $Y(t) \in VX_0(tr_0)$ for all $t \in [-n, n]$. Let U be an open identity neighborhood of G such that $UU \subseteq V$. Let $\delta \in]0, 1]$ be such that $|r - r_0| < \delta$ implies $X_0(tr) \in UX_0(tr_0)$ for $t \in [-n, n]$. Find $m \in \mathbb{N}$ so that $|r - r_0| \leqslant 1$ and $|t| \leqslant n$ imply $|tr| \leqslant m$. Then $(r, X) \in]r_0 - \delta, r_0 + \delta[\times W(X_0; m, U)$ and $|t| \leqslant n$ imply $|tr| \leqslant m$ and thus

$$(r \cdot X)(t) = X(tr) \in UX_0(tr) \subseteq UUX_0(tr_0) \subseteq V(r_0 \cdot X_0)(t),$$

that is, $r \cdot X \in W(r_0 \cdot X_0; n, V)$.

The action $(r, X) \mapsto r \cdot X : \mathbb{R} \times \mathfrak{L}(G) \to \mathfrak{L}(G)$ will be called the *scalar multiplication* of $\mathfrak{L}(G)$.

2 The Lie algebra of a topological group

We define the *commutator* $\operatorname{comm}(g, h)$ of two elements g, h of a group to be $ghg^{-1}h^{-1}$.

For a topological group G and one parameter subgroups $X, Y \in \mathfrak{L}(G)$ we define continuous functions $XY, \operatorname{comm}(X, Y) : \mathbb{R} \to G$ by $(XY)(r) = X(r)Y(r)$ and $\operatorname{comm}(X, Y)(r) = \operatorname{comm}(X(r), Y(r))$. Of course these functions are not one parameter subgroups in general except when $\operatorname{comm}(X(r), Y(r)) = 1$ for all r, which is certainly the case if G is commutative.

For a subset A of a group G we shall write $\langle A \rangle$ for the subgroup algebraically generated by A in G.

Definitions 2.1. Let G be a topological group. Then it is said that *G has a Lie algebra* or, equivalently, that G is a *topological group with a Lie algebra* if the following conditions hold:

(i) For all $X, Y \in \mathfrak{L}(G)$, the following limits exist pointwise for all $t \in \mathbb{R}$:

$$(X + Y)(t) \stackrel{\text{def}}{=} \lim_{n \to \infty} (X(t/n)Y(t/n))^n, \tag{$*$}$$

$$[X, Y](t^2) \stackrel{\text{def}}{=} \lim_{n \to \infty} \operatorname{comm}(X(t/n), Y(t/n))^{n^2} \tag{$**$}$$

and $X + Y, [X, Y] \in \mathfrak{L}(G)$.

(ii) Addition $(X, Y) \mapsto X + Y : \mathfrak{L}(G) \times \mathfrak{L}(G) \to \mathfrak{L}(G)$ and bracket multiplication $(X, Y) \mapsto [X, Y] : \mathfrak{L}(G) \times \mathfrak{L}(G) \to \mathfrak{L}(G)$ are continuous.

(iii) With respect to scalar multiplication $\cdot$, addition $+$, and bracket multiplication $[\cdot, \cdot]$, the set $\mathfrak{L}(G)$ is a real Lie algebra. The Lie algebra $\mathfrak{L}(G)$ of a topological group is said to be *generating*, if the closed subgroup $\overline{\langle \exp \mathfrak{L}(G) \rangle}$ generated by the image is the identity component G_0 of G.

In particular, if G has a Lie algebra, then $\mathfrak{L}(G)$ is a *topological Lie algebra*. Note that a topological group G has a Lie algebra if and only if G_0 has a Lie algebra.

A real Lie group is a group object in the category of finite-dimensional smooth manifolds and smooth functions; this is but one of many possible equivalent definitions. However, any of the standard definitions of a (real) Lie group provides a finite-dimensional real Lie algebra $\mathfrak{L}(G)$ and an exponential function $\exp_G : \mathfrak{L}(G) \to G$ so that for every $x \in \mathfrak{L}(G)$ the function $r \mapsto \exp_G r \cdot x : \mathbb{R} \to G$ is a one parameter subgroup X of G such that $X(1) = \exp_G x$, and $x \mapsto X : \mathfrak{L}(G) \to \mathrm{Hom}(\mathbb{R}, G)$ is a homeomorphism. Thus we have

Theorem 2.2. *Every Lie group has a Lie algebra.*

Theorem 2.3. *Every abelian topological group has a Lie algebra, and its Lie algebra is commutative.*

Proof. If $X, Y \in \mathfrak{L}(G)$ and $\mathrm{comm}(X(r), Y(r)) = 0$ for all $r \in \mathbb{R}$, then

$$\left(X\left(\frac{t}{n}\right)Y\left(\frac{t}{n}\right)\right)^n = X(t)Y(t)$$

and thus 2.11(i) is clearly satisfied with $X + Y$ the pointwise product. Obviously the bracket product vanishes.

The proof of the following remark is a straightforward exercise:

Proposition 2.4. *If G is the additive group of a topological vector space E, then G is a topological group with Lie algebra, and $\mathfrak{L}(G) = E$ with zero bracket. Every topological vector space occurs in this fashion as the Lie algebra of a topological group.*

Proof. The function $\varphi : E \to \mathfrak{L}(G)$ defined by $\varphi(x)(t) = t \cdot x$ is a linear map which is the inverse of the exponential function $\exp : \mathfrak{L}(G) \to G$. Therefore $\mathfrak{L}(G)$ and E are isomorphic as topological vector spaces. In particular, every topological vector space occurs (up to isomorphism of topological vector spaces) as the Lie algebra of a topological group.

In [2, p. 337, Exercise E7.17] we have seen a closed arcwise connected subgroup G of the additive group of a separable Banach space E such that $\mathfrak{L}(G) = \{0\}$; in particular, G is an arcwise connected complete group without small subgroups which fails to be a Lie group. We also saw in [2, p. 135, Proposition 5.33(iv)] that a compact group without small subgroups is a Lie group; this remains true for locally compact groups, but the proof is much harder.

3 The category of topological groups with Lie algebras

We shall review and record the names of several categories of interest to us. In any category $\mathscr{A}$, the set of morphisms $A_1 \to A_2$ is denoted by $\mathscr{A}(A_1, A_2)$.

Definitions 3.1. (i) The *category of topological groups* (all of which are assumed to be Hausdorff) and continuous group homomorphisms as morphisms is denoted by $\mathbb{TOPGR}$.

(ii) If $\mathscr{C}$ is a category, a subcategory $\mathscr{A}$ is called *full* if $\mathscr{A}(A_1, A_2) = \mathscr{C}(A_1, A_2)$ for each pair of objects A_1, A_2 in $\mathrm{ob}(\mathscr{A})$. If C is any class of objects of a category $\mathscr{C}$, then $\mathscr{A} = \bigcup_{(A_1, A_2) \in C \times C} \mathscr{C}(A_1, A_2)$ is a full subcategory of $\mathscr{C}$ called the *full subcategory of C-objects.*

(iii) The full subcategory of $\mathbb{TOPGR}$ of all topological groups having a Lie algebra (in the sense of Definitions 2.1) is denoted by $\mathbb{LIEALGGR}$.

(iv) If X and Y are two topological spaces with scalar multiplication, then a function $f : X \to Y$ is called a *scalar morphism* if it is continuous and satisfies $f(t \cdot x) = t \cdot f(x)$ for all $t \in \mathbb{R}$ and $x \in X$. The category of Hausdorff topological spaces with scalar multiplication and scalar morphisms between them is denoted by $\mathbb{SCAL}$ (see discussion following 1.1).

It is worth taking note of the fact that a morphism $f : G \to H$ between two groups having a Lie algebra is simply a continuous group homomorphism.

Theorem 3.2. *Let $f : G \to H$ be a $\mathbb{TOPGR}$-morphism. Then there is a unique scalar morphism $\mathfrak{L}(f) : \mathfrak{L}(G) \to \mathfrak{L}(H)$ such that*

$$\begin{array}{ccc} \mathfrak{L}(G) & \xrightarrow{\mathfrak{L}(f)} & \mathfrak{L}(H) \\ {\scriptstyle \exp_G}\downarrow & & \downarrow{\scriptstyle \exp_H} \\ G & \xrightarrow[f]{} & H \end{array} \tag{1}$$

commutes. It is defined by $\mathfrak{L}(f)(X) = f \circ X$ for $X \in \mathfrak{L}(G)$.

Proof. Since f is continuous, if U is an identity neighborhood of G we find an identity neighborhood V of G such that $f(V) \subseteq U$; if now C is a compact subset of $\mathbb{R}$, then for $X, Y \in \mathfrak{L}(G)$, the relation $Y(r)^{-1}X(r) \in V$ for all $r \in C$ implies that

$$\mathfrak{L}(f)(Y)(r)^{-1}\mathfrak{L}(f)(X)(r) = (f \circ Y)(r)^{-1}(f \circ X)(r) = f(Y(r)^{-1}X(r)) \in f(V) \subseteq U$$

for all $r \in C$. It follows that $\mathfrak{L}(f)$ is continuous. Let $r \in \mathbb{R}$. Then

$$\begin{aligned} \mathfrak{L}(f)(r \cdot X)(t) &= f((r \cdot X)(t)) = f(X(tr)) = (f \circ X)(tr) \\ &= (r \cdot (f \circ X))(t) = (r \cdot \mathfrak{L}(f)(X))(t). \end{aligned}$$

Thus $\mathfrak{L}(f)$ is a scalar morphism. If $x \in \mathfrak{L}(G)$, then

$$\exp_H \mathfrak{L}(f)(X) = \mathfrak{L}(f)(X)(1) = f(X(1)) = f(\exp_G X).$$

Thus (1) is commutative. It remains to show uniqueness. Assume that

$$\Phi : \mathfrak{L}(G) \to \mathfrak{L}(H)$$

is a scalar morphism such that $\exp_H \circ \Phi = f \circ \exp_G$. Let $X \in \mathfrak{L}(G)$. Then

$$\exp_H(\Phi(r \cdot X)) = \exp_H(r \cdot \Phi(X)) = \Phi(X)(r)$$

on the one hand and $f(\exp_G r \cdot X) = f(X(r))$, that is, $\Phi(X) = f \circ X = \mathfrak{L}(f)(X)$.

Let $\mathbb{TOP}_*$ denote the category of pointed topological spaces and continuous maps preserving base-points, and let $I : \mathbb{SCAL} \to \mathbb{TOP}_*$ and $J : \mathbb{TOPGR} \to \mathbb{TOP}_*$ be the forgetful functors which assign to a space with scalar multiplication the underlying pointed space, respectively, to a topological group the underlying pointed space. A functor is said to be *continuous* if it preserves all limits. (See for instance [2, Appendix 3].)

Theorem 3.3. *The assignments* $G \mapsto \mathfrak{L}(G)$ *and* $f \mapsto \mathfrak{L}(f)$ *define a continuous functor* $\mathfrak{L} : \mathbb{TOPGR} \to \mathbb{SCAL}$ *and* $\exp : I \circ \mathfrak{L} \to J$ *is a natural transformation of functors*: $\mathbb{TOPGR} \to \mathbb{TOP}_*$.

Proof. We have to verify several statements.

(a) $\mathfrak{L}$ is a functor. If $f : G \to G$ is the identity map, then $\mathfrak{L}(f)(X) = f \circ X = X$ shows that $\mathfrak{L}(f)$ is the identity morphism. If $f_1 : G_1 \to G_2$ and $f_2 : G_2 \to G_3$ are morphisms of topological groups, then

$$\mathfrak{L}(f_2 \circ f_1)(X) = (f_2 \circ f_1) \circ X = f_2 \circ (f_1 \circ X) = (\mathfrak{L}(f_2) \circ \mathfrak{L}(f_1))(X).$$

Thus $\mathfrak{L}(f_2 \circ f_1) = \mathfrak{L}(f_2) \circ \mathfrak{L}(f_1)$.

(b) exp is a natural transformation $I \circ \mathfrak{L} \to J$. This is immediate from the definition of a natural transformation of functors in 1.1 and the commutativity of the diagram (1) above.

(c) The functor $\mathfrak{L}$ is continuous, i.e., preserves limits. It suffices to show that $\mathfrak{L}$ preserves products and equalizers. (See for instance [2, p. 722, Theorem A3.47].)

Products: Let $\{G_j \mid j \in J\}$ be a family of topological groups. Let

$$\mathrm{pr}_j : P \stackrel{\mathrm{def}}{=} \prod_{k \in J} G_k \to G_j$$

denote the projections. Define $\varphi : \mathfrak{L}(P) \to \prod_{j \in J} \mathfrak{L}(G_j)$ by $\varphi(X) = (\mathrm{pr}_j \circ X)_{j \in J}$; conversely, if $(X_j)_{j \in J} \in P$, then the universal property of the product gives a unique $X : \mathbb{R} \to P$ such that $X_j = \mathrm{pr}_j \circ X$. If we write $X = \psi((X_j)_{j \in J})$, then

$$\psi : \prod_{j \in J} \mathfrak{L}(G_j) \to P$$

is an inverse of φ. Since both φ and ψ preserve scalar multiplication (which is componentwise on products) we have

$$\mathfrak{L}\Big(\prod_{j\in J} G_j\Big) \cong \prod_{j\in J} \mathfrak{L}(G_j).$$

Equalizers: Let $f_1, f_2 : G \to H$ be two morphisms of topological groups, set $E = \{g \in G \mid f_1(g) = f_2(g)\}$ and let $e : E \to G$ be the inclusion. Then E is a closed subgroup of G and e is the equalizer of f_1 and f_2. We claim that $\mathfrak{L}(e) : \mathfrak{L}(E) \to \mathfrak{L}(G)$ is the equalizer of $\mathfrak{L}(f_1), \mathfrak{L}(f_2) : \mathfrak{L}(G) \to \mathfrak{L}(H)$. For this purpose, let $X \in \mathfrak{L}(E)$. Then

$$\mathfrak{L}(f_1)(X)(t) = f_1(X(t)) = f_2(X(t)) = L(f_2)(X)(t).$$

Thus X equalizes $\mathfrak{L}(f_1)$ and $\mathfrak{L}(f_2)$. Conversely assume that $X \in \mathfrak{L}(G)$ equalizes $\mathfrak{L}(f_1)$ and $\mathfrak{L}(f_2)$. Then

$$f_1(X(t)) = \mathfrak{L}(f_1)(X)(t) = \mathfrak{L}(f_2)(X)(t)$$

for all $t \in \mathbb{R}$ so that $X(t) \in E$ for all t, and thus $X \in \mathfrak{L}(E)$.

This concludes the proof that $\mathfrak{L}$ preserves products and equalizers and thus arbitrary limits.

To category theorists, the continuity of $\mathfrak{L}$ is not a surprise because

$$\mathfrak{L} = \mathrm{Hom}(\mathbb{R}, -) : \mathbb{TOPGR} \to \mathbb{SCAL}$$

is a hom-functor.

Definitions 3.4. Let G be a topological group and let G_a denote the arc component of 1 in G, called the *identity arc component* of G. Finally, let $E(G)$ denote the smallest closed subgroup of G containing the images of all one parameter subgroups; that is

$$E(G) = \overline{\langle \exp_G(\mathfrak{L}(G))\rangle}. \tag{2}$$

Observe that each of G_0, G_a and $E(G)$ is mapped into itself by any continuous map $G \to G$ which fixes 1 and thus by any endomorphism of G; that is, each of these three groups is a fully characteristic subgroup of G (cf. [2, p. 23]). Notice that G_a may not be closed as the example of the p-adic solenoid shows; see also [2, (1.28), (1.38)], and [2, Chapter 8] in general.

We say that the Lie algebra $\mathfrak{L}(G)$ is *generating* if G has a Lie algebra and $G_0 = E(G)$.

Since $\exp_G \mathfrak{L}(G)$ is arcwise connected, so is $\langle \exp_G \mathfrak{L}(G)\rangle$ and hence the closure of this group is connected. If G is a Lie group, then $\exp_G \mathfrak{L}(G)$ is an identity neighborhood, and this makes $\langle \exp_G \mathfrak{L}(G) \rangle$ an open (hence closed) subgroup, implying that $G_0 \subseteq E(G)$, and thus the Lie algebra of any Lie group is generating. The universe in which we shall work is that of topological groups which have a generating Lie algebra.

In [2, p. 337, Exercise E7.17] we encountered a closed contractible (and hence arcwise connected) non-singleton subgroup G of the additive Lie group of a Banach space which has no one parameter subgroups, and hence satisfies $E(G) = \{0\}$. If X is an arcwise connected compact pointed space and $F(X)$ is the free compact abelian group on X (see [2, p. 407ff.]) the subgroup $\langle X \rangle$ of $F(X)$ is free as an abelian group (see [2, p. 410, Proposition 8.52]) and thus as a topological group satisfies $E(\langle X \rangle) = \{0\}$ while being arcwise connected. Similar comments apply to the free (non-abelian) compact group. Thus there is an abundance of connected topological groups (even abelian ones) which have a Lie algebra due to the fact that they have no non-trivial one parameter subgroups. This confirms that no structural information via the exponential function is to be obtained unless the Lie algebra is generating.

If G has a Lie algebra, then the fully characteristic closed subgroup $E(G)$ has a generating Lie algebra by definition.

If H is a subgroup of a topological group G, then the coextension of any one parameter subgroup $X : \mathbb{R} \to H$ to G gives a one parameter subgroup $\mathbb{R} \to G$ which we shall also write as X by a slight abuse of notation; that is, if $\mathrm{incl} : H \to G$ is the inclusion, we identify X and $\mathrm{incl} \circ X$. Thus we write $\mathfrak{L}(H) \subseteq \mathfrak{L}(G)$.

The proof of the following proposition is straightforward from the definitions:

Proposition 3.5. *For any topological group G, one has*

$$\mathfrak{L}(\langle \exp_G \mathfrak{L}(G) \rangle) = \mathfrak{L}(G_a) = \mathfrak{L}(G_0) = \mathfrak{L}(G) = \mathfrak{L}(E(G)), \tag{3}$$

and the following statements are equivalent:

(i) *G has a Lie algebra*;

(ii) *G_0 has a Lie algebra*;

(iii) *$E(G)$ has a Lie algebra*;

(iv) *G_a has a Lie algebra*.

For locally compact groups G we have $G_a = \langle \exp \mathfrak{L}(G) \rangle$ and $G_0 = E(G)$, but that is not obvious. If we consider $G = \mathbb{Q}$, the additive group of rational numbers in the topology induced from that of $\mathbb{R} \supseteq \mathbb{Q}$, then $H = \mathbb{R}$ is the completion of G. By Theorem 2.3, both G and H have Lie algebras, and by Theorem 2.4 we have $\mathfrak{L}(H) = \mathbb{R}$. Thus $\mathfrak{L}(G) = \{0\} \neq \mathbb{R} = \mathfrak{L}(H)$. It follows that $\mathfrak{L}$ does not respect completion. Later we shall see a deeper reason: the functor $\mathfrak{L}$ is a right adjoint and the completion functor is a left adjoint.

In [2] it was shown that for any locally compact abelian group G one has $G_a = \exp \mathfrak{L}(G)$ (see [2, p. 389, Theorem 8.30(ii)]) and that for a compact abelian group

G, the factor group G/G_a, as an abstract abelian group, is isomorphic to $\mathrm{Ext}(\hat{G}, \mathbb{Z})$ (see *loc. cit.* (iii)). In the case of the p-adic solenoid $\mathbb{T}_p$ of Example 1.20(A)(ii) we have $\mathfrak{L}(\mathbb{T}_p) \cong \mathbb{R}$, and $\exp_{\mathbb{T}_p} : \mathfrak{L}(\mathbb{T}_p) \to \mathbb{T}_p$ is injective. Thus $(\mathbb{T}_p)_a$ is a copy of $\mathbb{R}$ endowed with a properly coarser topology. By the preceding proposition we have $\mathbb{R} \cong \mathfrak{L}(\mathbb{T}_p) = \mathfrak{L}((\mathbb{T}_p)_a)$. In particular, $(\mathbb{T}_p)_a$ is a topological group with a Lie algebra by Theorem 2.3, and its Lie algebra is $\mathbb{R}$.

It should be clear that, in the spirit of universal topological algebra, a *topological Lie algebra* is a real Lie algebra L which is at the same time a Hausdorff topological space such that scalar multiplication $\mathbb{R} \times L \to L$, addition $L \times L \to L$ and Lie bracket $[\cdot\,,\cdot] : L \times L \to L$ are continuous.

Definition 3.6. The category of topological Lie algebras and continuous Lie algebra morphisms is denoted by $\mathbb{LIEALG}$.

Proposition 3.7. (i) *The category* $\mathbb{LIEALG}$ *of topological Lie algebras is complete, that is, has all limits.*

(ii) *The functor* $\mathfrak{L} : \mathbb{TOPGR} \to \mathbb{SCAL}$ *maps the category* $\mathbb{LIEALGGR}$ *of topological groups with Lie algebras into the category* $\mathbb{LIEALG}$ *of topological Lie algebras.*

Proof. Part (i) is a straightforward exercise showing that $\mathbb{LIEALG}$ has products and equalizers.

(ii) If G is a topological group with a Lie algebra, then by Definition 2.11, $\mathfrak{L}(G) \in \mathrm{ob}(\mathbb{SCAL})$ is a topological Lie algebra and thus belongs to the subcategory $\mathbb{LIEALG}$. Now let $f : G \to H$ be a $\mathbb{LIEALGGR}$-morphism. We know that $\mathfrak{L}(f) : \mathfrak{L}(G) \to \mathfrak{L}(H)$ is a $\mathbb{SCAL}$-morphism. We must show that for $X, Y \in \mathfrak{L}(G)$ we have $f(X + Y) = f(X) + f(Y)$ and $f[X, Y] = [f(X), f(Y)]$.

Firstly we deal with addition. By 2.11(4) we have

$$(X + Y)(t) = \lim_{n\to\infty} (X(t/n)\,Y(t/n))^n.$$

Since f is continuous and a group morphism we have

$$\begin{aligned}\mathfrak{L}(f)(X + Y)(t) &= f((X + Y)(t)) = \lim_{n\to\infty} (f(X(t/n))f(Y(t/n)))^n \\ &= \lim_{n\to\infty} \left(\frac{1}{n} \cdot (f \circ X) \frac{1}{n} \cdot (f \circ Y)\right)^n (t) = (\mathfrak{L}(f)(X) + \mathfrak{L}(f)(Y))(t).\end{aligned}$$

Next we treat the Lie bracket. By 2.11(5) we have

$$[X, Y](t^2) = \lim_{n\to\infty} \mathrm{comm}(X(t/n), Y(t/n))^{n^2}.$$

Accordingly, we get this time

$$\begin{aligned}\mathfrak{L}(f)[X,Y](t^2) = f([X,Y](t)) &= \lim_{n\to\infty} \operatorname{comm}(f(X(t/n)), f(Y(t/n)))^{n^2}\\ &= \lim_{n\to\infty} \operatorname{comm}\left(\frac{1}{n}\cdot(f\circ X), \frac{1}{n}\cdot(f\circ Y)\right)^{n^2}(t^2)\\ &= [\mathfrak{L}(f)(X), \mathfrak{L}(f)(Y)](t^2).\end{aligned}$$

Thus $\mathfrak{L}(f)$ is a Lie algebra morphism, and the proof of the proposition is complete.

The largest category of topological groups which has a Lie theory is the full subcategory $\mathbb{LIEGR}$ of $\mathbb{TOPGR}$ of topological groups which have a Lie algebra. The next theorem shows that this is a complete category and that the Lie algebra functor defined on it preserves limits. In the end, the 'right' category for Lie theory is a complete full subcategory of $\mathbb{LIEGR}$, for which there are many candidates.

Theorem 3.8 (The Completeness Theorem of the Category of Groups with Lie Algebras).

(i) *The category* $\mathbb{LIEALGGR}$ *of topological groups having a Lie algebra is closed in the category* $\mathbb{TOPGR}$ *of topological groups under the formation of arbitrary limits and passage to closed subgroups. In particular*, $\mathbb{LIEALGGR}$ *is a complete category.*

The full subcategory $\mathbb{LIEALGGENGR}$ *of all topological groups have a generating Lie algebra is closed under the formation of arbitrary products and passage to retracts.*

(ii) *The functor* $\mathfrak{L}:\mathbb{LIEALGGR}\to\mathbb{LIEALG}$ *is continuous, i.e. preserves all limits.*

Proof. Once it is shown that $\mathbb{LIEALGGR}$ is closed under the formation of limits, from Theorem 3.3 and Proposition 3.7 we conclude that

$$\mathfrak{L}:\mathbb{LIEALGGR}\to\mathbb{LIEALG}$$

is a continuous functor. Thus, by [3], Theorem 1.11(ii) it suffices to show that $\mathbb{LIEALGGR}$ and $\mathbb{LIEALGGENGR}$ are closed under the formation of products and the passing to closed subgroups in $\mathbb{TOPGR}$.

Products: Let $\{G_j \mid j\in J\}$ be a family of topological groups and $P \overset{\text{def}}{=} \prod_{j\in J} G_j$ its product. From Theorem 3.2 we know that we can write

$$\mathfrak{L}\Big(\prod_{j\in J} G_j\Big) = \prod_{j\in J}\mathfrak{L}(G_j)$$

such that

$$\exp_P(X_j)_{j\in J} = (\exp_{G_j} X_j)_{j\in J}.$$

Since each $\mathfrak{L}(G_j)$ is a topological Lie algebra, so is $\prod_{j\in J}\mathfrak{L}(G_j)$. Moreover,

$$(X_j)_{j\in J} + (Y_j)_{j\in J} = (X_j + Y_j)_{j\in J} = \lim_{n\to\infty}\left(\left(\frac{1}{n}\cdot X_j \frac{1}{n}\cdot Y_j\right)^n\right)_{j\in J}$$
$$= \lim_{n\to\infty}\left(\frac{1}{n}\cdot (X_j)_{j\in J}\frac{1}{n}\cdot (Y_j)_{j\in J}\right)^n.$$

This shows that condition 2.11(i)(4) holds. Analogously one proves condition 2.11(i)(5). One observes straightforwardly that

$$\Big(\prod_{j\in J} G_j\Big)_0 = \prod_{j\in J}(G_j)_0 \quad \text{and} \quad E\Big(\prod_{j\in J} G_j\Big) = \prod_{j\in J} E(G_j)$$

and concludes readily from these facts that $\mathbb{LIEALGGENGR}$ is closed in $\mathbb{TOPGR}$ under the formation of products.

Passage to closed subgroups: Let H be a closed subgroup of G. We may assume that $\mathfrak{L}(H) \subseteq \mathfrak{L}(G)$, as every one parameter subgroup of H may be considered as one of G. The relations

$$(X+Y)(t) \overset{\text{def}}{=} \lim_{n\to\infty}(X(t/n)Y(t/n))^n, \tag{*}$$

$$[X,Y](t^2) \overset{\text{def}}{=} \lim_{n\to\infty}\operatorname{comm}(X(t/n),Y(t/n))^{n^2} \tag{**}$$

and $X+Y, [X,Y] \in \mathfrak{L}(H)$ hold as they hold in G and since H is closed. Thus H is a topological group with Lie algebra and thus belongs to $\mathbb{LIEALGGR}$.

Retracts: Let $p: G \to H$ be a retraction in $\mathbb{TOPGR}$, i.e. there is a morphism $j: H \to G$ in $\mathbb{TOPGR}$ such that $pj = \mathrm{id}_H$. There is no loss in generality in assuming that H is a subgroup of G and that $j: H \to G$ is the inclusion map (cf. E1.5), and since H is a retract, it is in fact a *closed* subgroup. Assume that G belongs to $\mathbb{LIEALGGENGR}$. Then H belongs to $\mathbb{LIEALGGR}$ by the preceding arguments. Functors preserve retractions and coretractions; specifically, $\mathfrak{L}(p): \mathfrak{L}(G) \to \mathfrak{L}(H)$ is a retraction with $\mathfrak{L}(p)\circ\mathfrak{L}(j) = \mathfrak{L}(pj) = \mathfrak{L}(\mathrm{id}_H) = \mathrm{id}_{\mathfrak{L}(H)}$. Thus $\mathfrak{L}(H)$ is a retract of $\mathfrak{L}(G)$. We notice that $p(E(G)) \subseteq E(H)$. Hence, if $G \in \operatorname{ob}\mathbb{LIEALGGENGR}$, then $p(G_0) = p(E(G)) \subseteq E(H)$. Now $p|H = pj = \mathrm{id}_H$, whence $H_0 = p(H_0) \subseteq p(G_0)$. Thus $H_0 \subseteq E(H) \subseteq H_0$, and equality follows. Thus $\mathbb{LIEALGGENGR}$ is closed in $\mathbb{TOPGR}$ under passing to retracts, and this completes the proof.

In the above proof we used the fact that *a retraction $p: G \to H$ of topological groups maps G_0 onto H_0*. This is a special situation for retractions. In general, even quotient maps between locally compact abelian groups fail to map components onto components. In [2, p. 19, E1.11] one finds a quotient morphism $\mathbb{Z}_p \times \mathbb{R} \to \mathbb{T}_p$; the identity component of the domain is $\{0\}\times\mathbb{R}$ and the p-adic solenoid $\mathbb{T}_p$ is connected. (This is only a special case of a general theorem on abelian topological groups: see [2, p. 379, Theorem 8.20].) We also recall that there are examples of additive groups of separable Banach spaces G having closed arcwise connected subgroups H with zero Lie algebra (cf. [2, p. 337, Exercise E7.17]). Then G is in $\mathbb{LIEALGGENGR}$ while

H is not. Since G and G/H are in $\mathbb{LIEALGGENGR}$ the subgroup H, as a kernel, is a limit. Thus the category $\mathbb{LIEALGGENGR}$ is not closed under the passage to all limits and thus fails to be complete.

The category $\mathbb{LIEGR}$ of Lie groups, which by Theorem 2.2 is a subcategory of $\mathbb{LIEALGGR}$, has finite products and equalizers, hence finite limits. Theorem 2.25 shows that *all* limits, in particular, arbitrary products and projective limits of Lie groups, are topological groups with Lie algebras.

4 The Lie Algebra functor has a left adjoint

We recall that a functor $U : \mathscr{A} \to \mathscr{B}$ is said to have a *left adjoint* $F : \mathscr{B} \to \mathscr{A}$ if there is a natural transformation $\eta_B : B \to U(F(B))$ such that for each morphism $f : B \to U(A)$ in $\mathscr{B}$ there is a unique morphism $f' : F(B) \to A$ such that $f = U(f') \circ \eta_B$. In diagram form:

$$
\begin{array}{ccccc}
\mathscr{B} & & & & \mathscr{A} \\
\hline
B & \xrightarrow{\eta_B} & U(F(B)) & & F(B) \\
{\scriptstyle\forall f}\big\downarrow & & \big\downarrow{\scriptstyle U(f')} & & \big\downarrow{\scriptstyle \exists! f'} \\
U(A) & \xrightarrow[\mathrm{id}_{U(A)}]{} & U(A) & & A
\end{array}
\qquad (\top)
$$

For details we refer to any text on category theory or to [2, Appendix 3]. The relation between the two adjoint functors may be expressed by saying that there is a natural bijection between the sets $\mathscr{B}(B, U(A))$ and $\mathscr{A}(F(B), A)$. In fact, in our notation, the bijection is implemented by the function $f \mapsto f'$ and its inverse

$$g \mapsto U(g) \circ \eta_B : \mathscr{A}(F(B), A) \to \mathscr{B}(B, U(A)).$$

The natural transformation η is called the *front adjunction*. There is, dually, a *back adjunction* $\pi_A : F(U(A))$ such that for each morphism $g : F(B) \to A$ there is a unique morphism $g' : B \to U(A)$ such that $g = \pi_A \circ F(g')$ (see for instance [2, p. 719, Proposition A3.36]).

$$
\begin{array}{ccccc}
\mathscr{B} & & & & \mathscr{A} \\
\hline
U(A) & & F(U(A)) & \xrightarrow{\pi_A} & A \\
{\scriptstyle\exists! g'}\big\uparrow & & {\scriptstyle F(g')}\big\uparrow & & \big\uparrow{\scriptstyle \forall g} \\
B & & F(B) & \xrightarrow[\mathrm{id}_{F(B)}]{} & F(B)
\end{array}
$$

Among the characteristic properties of adjunctions the following one will be relevant in the present context:

if B is an object in $\mathscr{B}$, then $\pi_{F(B)} \circ F(\eta_B) : F(B) \to F(B)$ is the identity map of the $\mathscr{A}$-object $F(B)$.

Similarly,

if A is an object of $\mathscr{A}$, then $U(\pi_A) \circ \eta_{U(A)} : U(A) \to U(A)$ is the identity map of the $\mathscr{B}$-object $U(A)$.

(See for instance [2, p. 719, Proposition A3.38].)

The crucial categorical theorem that we invoke is the Adjoint Existence Theorem. First we give the relevant definition:

Definition 4.1 (The Solution Set Condition). The functor $U : \mathscr{B} \to \mathscr{A}$ satisfies the *solution set condition* if for each $A \in \mathrm{ob}\,\mathscr{A}$ there is a *set* $S(A)$ of pairs (φ, M), $\varphi : A \to UM$ such that for every pair (f, B), $f : A \to UB$ there is some $(\varphi, M) \in S(A)$ with some factorization $f = (Uf_0)\varphi$ and $f_0 : M \to B$.

Now we can formulate one of the basic results in category theory:

Theorem 4.2 (The Adjoint Functor Existence Theorem). *Assume that $\mathscr{B}$ is a complete category. Then for a functor $U : \mathscr{B} \to \mathscr{A}$ the following conditions are equivalent*:

(1) *U has a left adjoint $F : \mathscr{A} \to \mathscr{B}$*;

(2) *U preserves limits and satisfies the solution set condition.*

Most sources on category theory present a proof; cf. also [2, p. 728, Theorem A3.60].

In checking the concrete occurrences of the situation of the Adjoint Functor Existence Theorem one observes that the Solution Set Condition practically never causes problems, and all the other conditions are readily verified, and we shall see this here.

In dealing with topological groups, we find the following a very useful first application of the Adjoint Functor Existence Theorem:

Theorem 4.3 (The Retraction Theorem for Full Closed Subcategories of $\mathbb{TOPGR}$). *For any full subcategory $\mathscr{G}$ of the category $\mathbb{TOPGR}$ of topological groups and continuous morphisms that is closed in $\mathbb{TOPGR}$ under the formation of products and under the passage to closed subgroups, there is a left adjoint functor*

$$F : \mathbb{TOPGR} \to \mathscr{G}$$

which agrees on $\mathscr{G}$ (up to a natural isomorphism) with the identity functor on $\mathscr{G}$. In particular given any topological group G, there exist a topological group FG in $\mathscr{G}$ and a morphism $\theta_G : G \to FG$ with dense image having the following universal property: *for every morphism $f : G \to H$ into a $\mathscr{G}$-group H there is a unique morphism $f' : FG \to H$ such that $f = f' \circ \theta_G$.*

Proof. The existence of F is immediate from Theorem 4.3 once we verify the solution set condition for the inclusion functor $\mathscr{G} \to \mathbb{TOPGR}$. Let G be a topological group. Let us say that morphisms $\varphi_j : G \to M_j$, $j = 1, 2$ in $\mathbb{TOPGR}$ are equivalent, if there is an isomorphism $\psi : M_1 \to M_2$ such that $\varphi_2 = \psi \circ \varphi_1$. From each equivalence class of morphisms $\varphi : G \to M$ such that $M = \bar{\varphi}(G)$ and M is a topological group in $\mathcal{G}$, let us pick one representative. Let us call the class of all such representatives $S(G)$. Then $S(G)$ is a set, because there is, up to equivalence, only a set of images $\varphi(G)$ under any morphism φ with domain G and a set of topologies on each $\varphi(G)$ and that there is up to a natural equivalence (in an obvious sense) a set of Hausdorff topological spaces in which a topological space $\varphi(G)$ is dense, since the cardinal of such a space is not bigger than the cardinal of the set of all filters on $\varphi(G)$. Now let $f : G \to H$ be a morphism from G to a topological group H in $\mathcal{G}$. Since $\mathcal{G}$ is closed under passing to closed subgroups, $\overline{f(G)}$ belongs to this category. Thus the corestriction $f' : G \to \overline{f(G)}$ of f to the closure of its image has an equivalent representative in $S(G)$. For brevity we may assume that $f' \in S(G)$. Let $f_0 : \overline{f(G)} \to H$ be the inclusion map. Then $f = f_0 \circ f'$. Thus the solution set condition for the inclusion functor $\mathcal{G} \to \mathbb{TOPGR}$ is verified.

It remains to verify that θ_G has dense image; but that is immediate from the assumption that $\mathcal{G}$ is closed under passing to closed subgroups and that the corestriction $G \to \overline{\theta_G(G)}$ of θ_G to the closure of its image has the universal property of θ_G and thus must agree with θ_G.

$$\begin{array}{ccccc}
\multicolumn{3}{c}{\mathbb{TOPGR}} & & \mathcal{G}\\ \hline
G & \xrightarrow{\theta_G} & FG & & FG\\
\forall f\big\downarrow & & \big\downarrow f' & & \big\downarrow \exists! f'\\
H & \xrightarrow[\mathrm{id}_H]{} & H & & H
\end{array}$$

A typical example for $\mathcal{G}$ is the category $\mathbb{COMPGR}$ of compact groups. In that case for each topological group G, the compact group $\alpha(G) \stackrel{\mathrm{def}}{=} FG$ is the *Bohr compactification* of G and $\theta_G : G \to \alpha(G)$ is the *Bohr compactification morphism*.

Another example is the category $\mathbb{ABTOPGR}$ of topological abelian groups. Then FG is the commutator factor group $G/\overline{G'}$ where G' is the algebraic commutator group.

A further relevant example is this:

Corollary 4.4. *Let $\mathbb{CTOPGR}$ denote the full subcategory of complete topological groups. Then there is a functor $G \mapsto G^* : \mathbb{TOPGR} \to \mathbb{CTOPGR}$ such that given any topological group G, there exist a complete topological group G^* and a morphism $\kappa_G : G \to G^*$ with dense image, such that for every morphism $f : G \to H$ into a complete topological group H there is a unique morphism $f' : G^* \to H$ such that $f = f' \circ \kappa_G$.*

$$\begin{array}{ccccc}
\multicolumn{3}{c}{\mathbb{TOPGR}} & & \mathbb{CTOPGR}\\ \hline
G & \xrightarrow{\kappa_G} & G^* & & G^*\\
\forall f\big\downarrow & & \big\downarrow f' & & \big\downarrow \exists! f'\\
H & \xrightarrow[\mathrm{id}_H]{} & H & & H
\end{array}$$

Proof. This is an immediate consequence of the completeness of the subcategory $\mathbb{CTOPGR}$ and Theorem 4.3.

One should remember that κ will not always be an embedding. We say that a topological group G *has a completion* if $\kappa_G : G \to G^*$ is an embedding, in which case we like to consider G as a dense subgroup of G^*.

The final appropriate example in our situation is the following:

Corollary 4.5. *There is a functor $\lambda : \mathbb{TOPGR} \to \mathbb{LIEALGGR}$ such that given any topological group G, there exist a topological group $\lambda(G)$ with Lie algebra and a morphism $\theta_G : G \to \lambda(G)$ with dense image, such that for every morphism $f : G \to H$ into a topological group H with Lie algebra, there is a unique morphism $f' : \lambda(G) \to H$ such that $f = f' \circ \theta_G$.*

$$\begin{array}{ccc|c}
\multicolumn{3}{c}{\mathbb{TOPGR}} & \mathbb{LIEALGGR} \\
\hline
G & \xrightarrow{\theta_G} & \lambda(G) & \lambda(G) \\
\forall f \downarrow & & \downarrow f' & \downarrow \exists! f' \\
H & \xrightarrow[\mathrm{id}_H]{} & H & H
\end{array}$$

Proof. This is an immediate consequence of the completeness of the subcategory $\mathbb{LIEALGGR}$ and Theorem 4.3.

The major application of the Adjoint Functor Existence Theorem, however in the present context is the following:

Theorem 4.6 (The Adjunction Theorem for $\mathfrak{L}$). (i) *The functor*

$$\mathfrak{L} : \mathbb{LIEALGGR} \to \mathbb{LIEALG}$$

has a left adjoint

$$\Gamma : \mathbb{LIEALG} \to \mathbb{LIEALGGR}.$$

In other words, the following assertions hold:

(i′) *For each topological Lie algebra $\mathfrak{g}$ there is a functorially associated topological group $\Gamma(\mathfrak{g})$ with a Lie algebra and there is a natural transformation $\eta_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{L}(\Gamma(\mathfrak{g}))$ such that for each morphism $f : \mathfrak{g} \to \mathfrak{L}(H)$ of topological Lie algebras, there is a unique morphism $f' : \Gamma(\mathfrak{g}) \to H$ such that $f = \mathfrak{L}(f') \circ \eta_{\mathfrak{g}}$. In diagram form,*

$$\begin{array}{ccc|c}
\multicolumn{3}{c}{\mathbb{LIEALG}} & \mathbb{LIEALGGR} \\
\hline
\mathfrak{g} & \xrightarrow{\eta_{\mathfrak{g}}} & \mathfrak{L}(\Gamma(\mathfrak{g})) & \Gamma(\mathfrak{g}) \\
\forall f \downarrow & & \downarrow \mathfrak{L}(f') & \downarrow \exists! f' \\
\mathfrak{L}(H) & \xrightarrow[\mathrm{id}_{\mathfrak{L}(H)}]{} & \mathfrak{L}(H) & H
\end{array} \qquad (\top)$$

(i$''$) *Let G be a group with a Lie algebra. Set $\tilde{G} \stackrel{\text{def}}{=} \Gamma(\mathfrak{L}(G))$. Then there is a natural transformation $\pi_G : \tilde{G} \to G$ such that for each topological Lie algebra $\mathfrak{g}$ and each morphism $f : \Gamma(\mathfrak{g}) \to G$ there is a unique morphism $f' : \mathfrak{g} \to \mathfrak{L}(G)$ of topological Lie algebras such that $f = \pi_G \circ \Gamma(f')$.*

$$\begin{array}{c|ccc}
\mathbb{LIEALG} & & \mathbb{LIEALGGR} & \\
\hline
\mathfrak{L}(G) & \tilde{G} & \xrightarrow{\pi_G} & G \\
{\scriptstyle \exists! f'}\big\uparrow & {\scriptstyle \Gamma(f')}\big\uparrow & & \big\uparrow{\scriptstyle \forall f} \\
\mathfrak{h} & \Gamma(\mathfrak{h}) & \xrightarrow[\mathrm{id}_{\Gamma(\mathfrak{h})}]{} & \Gamma(\mathfrak{h})
\end{array} \qquad (\perp)$$

(ii) *The group $\Gamma(\mathfrak{g})$ has a generating Lie algebra and is therefore a member of* $\mathbb{LIEALGGENGR}$. *Thus Γ maps* $\mathbb{LIEALG}$ *into* $\mathbb{LIEALGGENGR}$.

(iii) *For a topological Lie algebra $\mathfrak{g}$, abbreviate $\Gamma(\mathfrak{g})$ to G. Then there are two inverse isomorphisms*

$$\pi_G : \tilde{G} \to G \quad \textit{and} \quad \Gamma(\eta_{\mathfrak{g}}) : G \to \tilde{G}.$$

Proof. (i) The proof is almost pure category theory; we will use the core existence result for adjoint functors which we have already invoked in the proof of Theorem 4.3. By the Left Adjoint Existence Theorem (see for instance [2, Appendix 3, p. 728, Theorem A3.60]), we have to verify that $\mathfrak{L}$ satisfies the Solution Set Condition 3.1, which in our case reads as follows: for each topological Lie algebra $\mathfrak{g}$, there is a *set* $S(\mathfrak{g})$ (and not a proper class) of pairs (f, H), where $f : \mathfrak{g} \to \mathfrak{L}(H)$ is a morphism of topological Lie algebras, such that for any pair (F, K), $F : \mathfrak{g} \to \mathfrak{L}(K)$, there exist a pair $(f, H) \in S(\mathfrak{g})$ and a morphism $f_0 : H \to K$ such that $F = \mathfrak{L}(f_0) \circ f$.

As is usual in such a situation, this condition is verified by establishing cardinality estimates.

(a) There is, up to equivalence, only a set of homomorphic surjective homomorphisms $f : \mathfrak{g} \to \mathfrak{h}$ of topological Lie algebras since there are cardinality bounds on the set of closed ideals $\mathfrak{i}$ of $\mathfrak{g}$ and the set of topologies on each quotient $\mathfrak{g}/\mathfrak{i}$.

(b) Given a Hausdorff topological space T there is a cardinality bound on all equivalence classes of dense embeddings of T into some Hausdorff space $\tilde{T}$, because there is a cardinality bound on the set of all filters on T, and because every point in a space $\tilde{T}$, in which T is contained densely, is the limit of a filter on T.

(c) Given a topological Lie algebra $\mathfrak{h}$, there is, up to equivalence, only a set of continuous functions $e : \mathfrak{h} \to S$ onto a Hausdorff space S up to equivalence; next, there is for each space S, up to isomorphism, at most a set of groups H which are algebraically generated by S, and there is at most a set of group topologies on H. Hence there is at most a set of topological groups H which have $\mathfrak{h}$ as their Lie algebra and satisfy $H = \langle \exp_H \mathfrak{h} \rangle$. Moreover, by (b) above, there is at most a set of Hausdorff topological groups in which $\langle \exp_H \mathfrak{h} \rangle$ is dense.

We say that two pairs (f_j, H_j), $f_j : \mathfrak{g} \to \mathfrak{L}(H_j)$, $j = 1, 2$ are equivalent if there is an isomorphism $\varphi : H_1 \to H_2$ such that $f_2 = \mathfrak{L}(\varphi) \circ f_1$. Now we consider the class of all pairs (f, H), $f : \mathfrak{g} \to \mathfrak{L}(H)$ such that $H = \overline{\langle \exp_H(f(\mathfrak{g})) \rangle}$. The preceding cardinality considerations show that we have a *set* $S(\mathfrak{g})$ of representatives (f, H) for the equivalence classes of such pairs.

Now let K be a topological group with a Lie algebra and $F : \mathfrak{g} \to \mathfrak{L}(K)$ a morphism of topological Lie algebras. Let $H = \overline{\langle \exp_K f(\mathfrak{g})\rangle}$. Then H is a closed subgroup of K and thus has a Lie algebra by Theorem 3.8(i). Moreover, the corestriction $f : \mathfrak{g} \to \mathfrak{L}(H)$ gives a pair (f, H) which is equivalent to a member of $S(\mathfrak{g})$. If $f_0 : H \to K$ is the inclusion morphism then $F = \mathfrak{L}(f_0) \circ f$, and since (f, H) is equivalent to a member of $S(\mathfrak{g})$, this proves that $\mathfrak{L}$ satisfies the Solution Set Condition. Since $\mathfrak{L}$ is continuous by Theorem 3.3, the Left Adjoint Functor Existence Theorem applies and proves the existence of a left adjoint functor Γ for $\mathfrak{L}$.

(i′) and (i″): The universal properties expressed in (i′) and (i″) are equivalent and express the fact that Γ is a left adjoint of $\mathfrak{L}$. See for instance [2, p. 719, Proposition A3.36].

(ii) This assertion is a consequence of the selection of the solution set and the construction of the left adjoint functor from the solution set in the Left Adjoint Existence Theorem; indeed, $(\eta_{\mathfrak{g}}, \Gamma(\mathfrak{g}))$ is a member of the solution set $S(\mathfrak{g})$.

(iii) Let $\mathfrak{g}$ be a topological Lie algebra and set $G \stackrel{\text{def}}{=} \Gamma(\mathfrak{g})$. By (ii) we have $G = \overline{\langle \exp_G \eta_{\mathfrak{g}}(\mathfrak{g})\rangle}$. Now we set $\tilde{G} \stackrel{\text{def}}{=} \Gamma(\mathfrak{L}(G))$ and note that in a similar vein, we have $\tilde{G} = \overline{\langle \exp_{\tilde{G}} \eta_{\mathfrak{L}(G)}(\mathfrak{L}(G))\rangle}$. The situation is described by the following diagram:

$$\begin{array}{ccccc} \mathfrak{g} & \xrightarrow{\eta_{\mathfrak{g}}} & \mathfrak{L}(G) & \xrightarrow{\eta_{\mathfrak{L}(G)}} & \mathfrak{L}(\tilde{G}) \\ & & {\scriptstyle \exp_G}\big\downarrow & & \big\downarrow{\scriptstyle \exp_{\tilde{G}}} \\ & & G & \xrightarrow[\Gamma(\eta_{\mathfrak{g}})]{} & \tilde{G}. \end{array}$$

Thus $\langle \exp_{\tilde{G}} \eta_{\mathfrak{L}(G)}(\mathfrak{L}(G))\rangle = \Gamma(\eta_{\mathfrak{g}})\langle(\exp_G \mathfrak{L}(G))\rangle$ is dense in $\Gamma(\eta_{\mathfrak{g}})(G)$ on the one hand and in $\tilde{G}$ on the other. Hence $\Gamma(\eta_{\mathfrak{g}})$ has a dense image. We recall from our review preceding the theorem that $\pi_{\Gamma(\mathfrak{g})} \circ \Gamma(\eta_{\mathfrak{g}}) : \Gamma(\mathfrak{g}) \to \Gamma(\mathfrak{g})$ is the identity of $\Gamma(\mathfrak{g}) = G$, that is, $\pi_G \circ \Gamma(\eta_{\mathfrak{g}}) = \mathrm{id}_G$. We saw that the coretraction $\Gamma(\eta_{\mathfrak{g}})$ has a dense image; but then it must be surjective and thus an isomorphism whose inverse is π_G.

According to this Theorem we have a natural bijection between the sets

$$\mathbb{LIEALG}(\mathfrak{g}, \mathfrak{L}(G)) \quad \text{and} \quad \mathbb{LIEALGGR}(\Gamma(\mathfrak{g}), G).$$

One may wish to consider the adjunction theorems 4.4, 4.5 and 4.6 side by side. If G is a topological group, then there is a left reflection $\lambda(G)$, that is, a universally attached topological group with a Lie algebra; its Lie algebra $\mathfrak{L}(\lambda(G))$ is the image in $\mathbb{LIEALG}$ under a right adjoint, and $\Gamma(\mathfrak{L}(\lambda(G)))$ is the image of it under the left adjoint Γ. We have natural maps

$$\begin{array}{ccc} & & G \\ & & \big\downarrow{\scriptstyle \theta_G} \\ \Gamma(\mathfrak{L}(\lambda(G))) & \xrightarrow{\pi_G} & \lambda(G). \end{array}$$

On the lower level we are dealing with topological groups with Lie algebras, and the left lower corner is a group in $\mathbb{LIEALGGENGR}$. The completion functor $(\cdot)^*$ can be combined with Γ, giving for each topological Lie algebra $\mathfrak{g}$ in a functorial fashion a complete group $\mathbf{\Gamma}(\mathfrak{g}) \stackrel{\text{def}}{=} \Gamma(\mathfrak{g})^*$ with Lie algebra. By abuse of notation, one frequently writes Γ in place of $\mathbf{\Gamma}$ and trusts that the context makes it clear what is meant.

5 Sophus Lie's Third Fundamental Theorem

Theorem 4.6(i″) is a very general form of Lie's Third Theorem. This becomes more evident if one restricts one's attention to the class of topological Lie algebras, for which $\eta_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{L}(\Gamma(\mathfrak{g}))$ is an isomorphism. In that case $\Gamma\mathfrak{g}$ realizes a group whose Lie algebra is the given Lie algebra $\mathfrak{g}$. Lie algebras which are projective limits of finite-dimensional ones are called *pro-Lie algebras*. In [3, Chapter 6] we show that *for all pro-Lie algebras* $\mathfrak{g}$ *the morphism* $\eta_{\mathfrak{g}}$ *is an isomorphism.*

The full subcategory pro$\mathbb{LIEGR}$ of $\mathbb{TOPGR}$ consisting of all pro-Lie groups is closed under all limits and under passage to closed subgroups in $\mathbb{TOPGR}$. It is contained in $\mathbb{LIEALGGR}$. The Lie algebras of pro-Lie groups are pro-Lie algebras; writing pro$\mathbb{LIEALG}$ for the full subcategory of $\mathbb{LIEALG}$ of all pro-Lie algebras, we obtain in [3, Chapters 6, 8] the following

Corollary 5.1. *The pro-Lie algebra functor* $\mathfrak{L} : \text{pro}\mathbb{LIEGR} \to \text{pro}\mathbb{LIEALG}$ *has a left adjoint pro-Lie group functor* $\Gamma : \text{pro}\mathbb{LIEALG} \to \text{pro}\mathbb{LIEGR}$ *such that* $\eta_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{L}\Gamma(\mathfrak{g})$ *is an isomorphism for all pro-Lie algebras* $\mathfrak{g}$. *For each pro-Lie group* G *the pro-Lie group* $\tilde{G} \stackrel{\text{def}}{=} \Gamma\mathfrak{L}(G)$ *is simply connected, and whenever the underlying space of* G *has a universal covering space, then the back adjunction* $\pi_G : \tilde{G} \to G$ *is the universal covering morphism.*

Thus by virtue of the left adjoint of the Lie algebra functor, pro-Lie groups satisfy a perfect version of Lie's Third Theorem and have a generalization of the universal covering group even when a covering group does not exist in the topological sense.

There are Banach Lie algebras $\mathfrak{g}$ for which $\eta_{\mathfrak{g}}$ fails to be an isomorphism. In [3] it is shown that $\tilde{G}$ is homeomorphic to a product of $\mathbb{R}^I$ for some set I and $\prod_{j\in J} S_j$ for some family of simply connected simple Lie groups S_j.

6 Conclusion

The basic tools in classical Lie theory are the Lie algebra and the exponential map; this is the case even when we are well outside the class of traditional Lie groups. In [2] we showed that Lie theory applies to all compact groups and all locally compact abelian groups; but it is true that it applies to all locally compact groups. However, the category $\mathbb{LCGR}$ of all locally compact groups and continuous homomorphisms is not complete as it fails to have arbitrary products. So the category we seek should be better behaved than $\mathbb{LCGR}$ and should contain at least all connected locally compact groups. In [4] we showed that the category of pro-Lie groups is a complete subcategory of $\mathbb{TOPGR}$ and contains all connected locally compact groups.

Here we have shown that the full category $\mathbb{LIEALGGR}$ of all topological groups having Lie algebras is closed in the category $\mathbb{TOPGR}$ of all topological groups, that the property of having a Lie algebra is preserved under passage to closed subgroups, and that the functor $\mathfrak{L} : \mathbb{LIEALGGR} \to \mathbb{LIEALG}$ assigning to a topological group having a Lie algebra its topological Lie algebra, has a left adjoint functor $\Gamma : \mathbb{LIEALG} \to \mathbb{LIEALGGR}$ which attaches to a topological Lie algebra $\mathfrak{g}$ in a universal fashion a topological group $G \stackrel{\text{def}}{=} \Gamma(\mathfrak{g})$. The universal property guarantees that there is a natural morphism $\eta_\mathfrak{g} : \mathfrak{g} \to \mathfrak{L}(G)$ such that for any morphism $f : \mathfrak{g} \to \mathfrak{L}(H)$ of topological Lie algebras there is a unique morphism $f' : G \to H$ satisfying $f = \mathfrak{L}(f') \circ \eta_\mathfrak{g}$. For a topological group G with Lie algebra this provides functorially a topological group $\tilde{G} \stackrel{\text{def}}{=} \Gamma(\mathfrak{L}(G))$ and a morphism $\pi_G : \tilde{G} \to G$ such that any morphism $f : \Gamma(\mathfrak{g}) \to G$ for some topological Lie algebra $\mathfrak{g}$ factors through π_G in the form $f = \pi_G \circ \Gamma(f')$ for a unique morphism $f' : \mathfrak{g} \to \mathfrak{L}(G)$ of topological Lie algebras. Ostensibly, $\tilde{G}$ is a vast generalization of a 'universal covering group' of G and π_G a generalization of a universal covering morphism. But no topological hypotheses are needed here for the construction of $\tilde{G}$. All that is required from G is that it have a Lie algebra.

If $\mathfrak{g}$ is a pro-Lie algebra, that is, a projective limit of finite-dimensional Lie algebras, then $\eta_\mathfrak{g} : \mathfrak{g} \to \mathfrak{L}(\Gamma(\mathfrak{g}))$ is an isomorphism of topological Lie algebras; we may 'identify' the Lie algebra of $\Gamma(\mathfrak{g})$ with $\mathfrak{g}$. The existence of the functor Γ is therefore *the* appropriate expression of Lie's Third Fundamental Theorem saying that every pro-Lie algebra is realized as the Lie algebra of a pro-Lie group and how this is done in a functorially satisfactory fashion is explained in even greater generality in this paper.

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