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The structure of compact groups.
A primer for the student—a handbook for the expert.
(English)
[B] de Gruyter Studies in Mathematics, 25.
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The title and sub-title advertize fairly the authors' accomplishments: those who think they know it all will be pleasantly surprised to find much new, and the beginner may plunge in profitably at any of several potential starting points, depending on need and interest.

The list of Contents, covering five pages, is detailed and descriptive. An abbreviated version restricted to Chapter Titles reads as follows:
Chapter 1, Basic Topics and Examples.
Chapter 2, The Basic Representation Theory of Compact Groups.
Chapter 3, The Ideas of Peter and Weyl.
Chapter 4, Characters.
Chapter 5, Linear Lie Groups.
Chapter 6, Compact Lie Groups.
Chapter 7, Duality for Abelian Topological Groups.
Chapter 8, Compact Abelian Groups.
Chapter 9, The Structure of Compact Groups.
Chapter 10, Compact Group Actions.
Chapter 11, The Structure of Free Compact groups.
Chapter 12, Cardinal Invariants of Compact Groups.
Following these 12 Chapters are some 165 pages devoted to four Appendices, each a self-contained introduction to basic material (but culminating in certain cases in highly non-trivial content), as follows:
Appendix 1, Abelian Groups.
Appendix 2, Covering Spaces and Groups.
Appendix 3, A Primer of Category Theory.
Appendix 4, Selected Results in Topology and Topological Groups.

Despite their broad, descriptive nature, those brief lists hardly do justice to the scope of this wide-ranging monograph. The following sample chosen *via* a random walk through the Index gives a (still quite incomplete) indication of some of the tools and topics treated and developed in detail. (We omit here issues touched upon only *en passant.*)

Continuous cross sections; (arc) components of a group; "dimension" functions of many kinds, including cohomological dimension; Hopf algebras; fiber bundles; feebly complete topological vector spaces; isotropy groups; the (exact) Hom-Ext sequence; projectives and injectives; semi-direct products and factors, topological and algebraic; universal covers.

Despite the diversity of content suggested by the list above, inexorably and repeatedly the authors revert to the principal mission: to describe the *structure* of compact groups. (Hence, for example, Hilbert's problem is solved only for compact groups, and the name A. M. GLEASON does not appear in the list of References.) The pace is measured, neither rushed nor pedantic. The writing is generally crisp and direct, occasionally informal and even witty. The authors' collective voice and personality shines through as authoritative, sure-footed and self-assured. The reader is convinced of the truth of the unspoken message heard clearly: "This is the right way to do it."

There are a few idiosyncratic departures from standard or expected procedure. For example: A linear Lie group is (defined to be) a particular subgroup of the multiplicative group of a certain Banach algebra; a compact Lie group is (defined to be) a compact group G satisfying one of seven conditions shown equivalent, *e.g.*, (3) G has a faithful finite dimensional representation, and (7) G has no small normal subgroups; a proof of Haar measure is selected which rests on Zorn's lemma and gives the desired existence and uniqueness not for all locally compact groups but for compact groups only; the remarkable sandwich-type theorems
$\{0,1\}^{wG} \twoheadrightarrow G \supseteq \{0,1\}^{wG}$, $G \twoheadrightarrow [0,1]^{wG}$
of Vilenkin, Kuz′minov, Šapirovskiĭ *et al.*, valid for all compact groups with infinite weight wG, are given only for compact semisimple connected G.

Though the volume of material organized and presented by the authors is prodigious, the reader is constantly well-oriented through the use of four effective and ever-present road-maps:
—(1) The above-mentioned table of Contents is detailed and thorough.
—(2) Significant theorems are named as well as numbered, and the names are used consistently in internal cross-referencing.
—(3) Each Chapter and Appendix begins with a brief essay designed to place the forthcoming content into context, and ends with a scholarly "Postscript" summarizing what has just been done, and how and why; literary florishes run under free reign in these valuable paragraphs, prejudices are vetted, and an air of informality sustains a refreshing spirit. These contributions are a welcome and unusual departure from the dry writing which characterizes most volumes of comparable scholarly weight.

—(4) The 34-page, two-column software-driven Index is complete and well-organized.

One cannot in this space adequately summarize a work of this magnitude. Further, substantial portions lie outside this reviewer's domain of competence. It is appropriate however to list a few spots where the present treatment seems particularly felicitous, interesting, or valuable. In several of these, and often elsewhere, this presentation is the first one available with full supporting background in book format.

- 5.52(iii): Every separable Lie subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g} = \mathrm{Ł}(G)$ (G a linear Lie group) is recovered in the form $\mathrm{Ł}(H)$ with $H := \langle \exp \mathfrak{h} \rangle$.
- 5.64, Van der Waerden's theorem: Every homomorphism from a compact Lie group G with $G_0 = G_0'$ to a compact group is continuous.
- 6.55, Gotô's theorem: Every element of a compact, connected, semisimple Lie group is a commutator.
- 6.87, Auerbach's theorem generalized: In a nonabelian compact connected Lie group the set
$\{(g, h) \in G \times G : \langle g, h \rangle \text{ is dense and free}\}$
is dense in $G \times G$.
- 6.74, D. H. Lee's theorem: Every compact Lie group G contains a finite group E such that $G = G_0 E$ and $G_0 \cap E$ is central in G_0.
- 6.95 and 9.59: The only spheres which support a group structure are the spheres $S^0 = \{0, 1\}$, the circle $S^1 = \mathrm{SO}(2)$, and $S^3 = \mathrm{SU}(2)$.
- 8.30 and 8.62: The kernel and the range of the map exp (for G LCA) are respectively the arc component of the identity and the first homotopy group $\pi_1(G)$.
- 9.32: In a compact, connected group G, every maximal pro-torus is a maximal abelian subgroup, every two are conjugate, the union is G, and the intersection is $Z(G)$.
- 11.19: The free group $F(X)$ and the free abelian group $F_{\mathrm{ab}}(X)$ over a compact space are connected by the relation $F(X) = F_{\mathrm{ab}}(X) \times F'(X)$ iff $H^1(X, \mathbb{Z})$ is divisible, in particular if X is contractible.
- 11.17: $w(FX) = (wX)^{\aleph_0}$, for X compact.
- A1.70: A complete proof, including the necessities from logic and a thorough introduction to stationary sets and Jensen's principle $\diamond$, of the undecidability in ZFC of Whitehead's Problem: Is every uncountable abelian group A such that $\mathrm{Ext}(\mathrm{A}, \mathbb{Z}) = 0$ necessarily free? Inevitably, this treatment depends in part on
- A1.81 Shelah's Compactness Theorem: [ZFC] If in an abelian group A every subgroup B such that $|B| < |A|$ is free, then A itself is free.

A fair description of this book must emphasize its success as a *Handbook*.

With the unexplained exception of the construction of a partition of unity subordinate to an open cover in the context of paracompactness (no proof is provided, and no reference is made to relevant works of J. Dieudonné, J. Dugundji or E. A. Michael), the authors develop essentially every necessary tool and argument from scratch. Thus their adoption of the oft-abused expression "self-contained" to describe their work is legitimate. Despite the breadth of the 265-item list of References, however, honestly described in the Preface as "substantial but by ... no means exhaustive", the historically minded reader might wish for more generous and detailed attention throughout the book to original sources. Too often the authors take the easy way out, referring effortlessly to reliable old favorites such as the monographs of N. Bourbaki or R. Engelking; and some of those 265 References are not keyed to a specific result in the book, but are simply cited without explanation in an unadorned list titled Additional Reading at the end of the appropriate chapter.

The high degree of reliability and accuracy of this book is due in part to the fact that according to the authors' Preface it is the culmination of classroom notes and lecture notes which they have developed singly and together over a 20-30-year period. Surely the authors profited also from an additional jump-start on those substantial portions of the material treated here which originated in their own earlier published articles. (In this connection one thinks for example of Hofmann's work with P. S. Mostert on cohomology theories for compact abelian groups; most of the results relating to the free compact groups $F(X)$ and $F_{\mathrm{ab}}(X)$ and their structure, developed by the authors in a sequence of papers appearing in the years 1986–1991; and much of the material on cardinal invariants associated with compact groups, including the so-called *generating rank*.) In any case the reader acquainted with the high standards of scholarship and attention to detail which characterizes other works by the authors will be pleased but not surprised at the level of care and scientific integrity achieved here. Fledgling students in need of a *Primer* will, under its tutelage, mutate gently into Experts for whom a *Handbook* is appropriate. This book is built to last. One expects this pattern to repeat over many student-generations.

[W. W. Comfort (Middletown)]