Semigroup Forum 60 (2000) 1–3

The Structure of Compact Groups

A Primer for the Student - A Handbook for the Expert

by Karl H. Hofmann and Sidney A. Morris

De Gruyter Studies in Mathematics, 25, Berlin, New York, 1998, xviii+835 pp.

ISBN 3-11-0152681, US$149.95

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Communicated by Boris M. Schein

As this is the only 1990s' English language treatise known to the reviewer which covers the structure theory of compact groups, no reference is made in this review to other earlier specific works. It is assumed that interested readers will by now be familiar with those earlier works.

In the present treatise almost all of the classical areas are covered eg representation theory, duality theory. One exception being the work leading to Dynkin diagrams which the authors felt to be well and truly covered elsewhere. Also, the work devoted to profinite groups (projective limits of finite groups) is restricted to that needed for other parts, since the detailed investigation of such groups was too specialised away from the main thrust of the book. Indeed, the statement is made at one point that if the book were mainly about one thing it would be connected compact groups. Various aspects of cohomology are also included.(See later remarks about compact semigroups.)

A notable feature of the book is that the development eschews differential topology and geometry on manifolds in favour of extensive use of the exponential function on a Banach Algebra. Of course this then involves a deal of category theory and homology theory in a number of places. Early on it is established that a compact Lie group is a compact group with no small subgroups.

The development in many ways goes beyond that strictly needed just for compact groups. Often the approach is to develop in a general way then obtain the needed results for compact groups as specialisations. For example, the theory of linear Lie groups is extensively developed first. It is then proved that compact Lie groups are all linear Lie groups in which quotients are again compact Lie groups (hence linear again). The needed theorems for compact Lie groups are then derived from the linear Lie group theory.

Although, of course, extensive use is made of projective limits; the authors repeatedly emphasise that some of the most delicate aspects of compact group structure need to be obtained other than by such methods. They then carry out the promised other approaches in later chapters.

There is strong didactic emphasis on the need to settle the abelian cases then get at the general cases by using maximal (pro-) tori. The authors develop and make use of interesting facts (which they say are not widely reported) about the closure of commutator subgroups in compact Lie groups. Several references to Bourbaki are made, and it is stated that in many places Bourbaki has been influential. Avoidance of Stoke's Theorem on manifolds is cited as one important example.

The final Chapters 11 and 12 reflect the authors particular research interests and involve extensive work on free compact groups on the one hand and consideration of certain cardinal invariants for topological groups on the other. Sample results here are:

Theorem 11.19. *Let X be a compact connected pointed space. Then the free compact group (denoted FX) is the direct product of the free compact abelian group $F_{ab}X$ and the commutator subgroup $F'X$ if and only if $H^1(X, Z)$ is divisible.* ∎

Theorem 12.11. *Every compact group has a suitable set.* ∎

(For this second theorem we note that a suitable set S for a group G, is a subset of the group which is discrete, closed in $G\backslash\{1\}$ and for which the smallest closed subgroup of G containing S is G. The above theorem then enables the defining of the cardinal invariant $s(G)$ for a compact group G as the minimum cardinal possible for a suitable set for G.)

Compact semigroup theory is used at a few key places in the book. Some examples are:

1. Wendel's proof on the existence and uniqueness of Haar measure via the compact semigroup of Radon probability measures on a compact group.

2. The theorem of Hopf and Samelson on the cohomology Hopf algebra of a connected compact semigroup over a field of characteristic zero, observing that due to the structure of connected compact semigroups with identity, this has the same cohomology as a connected compact group.

3. At a number of other places the basic results about compact semigroups are applied and in the work on metrization in one of the appendices something is done on compact semigroups. The reader interested in semigroups may also wish to check pages 78, 517, 701 and 733ff.

The presentation of the book is first class with an easy to read typeface, an excellent set of 265 references and a well developed index. I noticed only a handful of misprints and they were obvious ones, such as an occasional repetition of words. There are a plenitude of exercises and examples, some integral to the development and some for additional interest.

The work is pretty much selfcontained as there are four Appendices dealing respectively with Abelian Groups, Covering Spaces and Groups, A Primer on Category Theory and Selected Results in Topology and Topological Groups. Also, any auxiliary needed material not covered by these appendices is developed on the spot.

A particularly pleasing and helpful feature of the presentation is the provision of Postscripts at the conclusion of each chapter. These not only summarise what each chapter is about but also provide roadmaps both forwards and backwards in the logical development.

Selected parts of a number of the chapters in the book taken together would serve well for advanced honours students or beginning graduate students as an introductory course in topological groups.

Other chapters, even in the general development, would tax all but the most persistent beginning graduate student and would better serve as an advanced seminar course or standby reference for advanced graduate students specialising in this area.

Researchers will be glad to have this treatise as a handy reference to dip into, and will be impressed by the clean development in a general topological algebra style, as a breath of fresh air compared to the traditional manifolds approach. One might say by way of analogy that much as a hydrofoil gets up on its skis and races away from other craft wallowing in the waves so does the whole approach here seek a high and speedy development after an initial surge of effort. This then enables one to perceive other vistas as well as getting the promised job done. Additionally the new material in Chapters 11 and 12 referred to above, opens up interesting new lines of research, some of which have already stimulated numbers of mathematicians to work successfully on them.

In conclusion, it is clear that in order to do full justice to this labour of love by two well known and respected mathematicians in the field, it would be necessary to work with it over a year or two so as to explore the extensive pool of ideas inherent in it and then write an in-depth review. That will I am sure take place in due course. However I will have done my job if I say at this stage merely that each mathematician interested in topological algebra should have a copy of this book on their shelf and make sure that their librarian gets one as well.

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