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FREE COMPACT GROUPS II: THE CENTER

Karl Heinrich HOFMANN
Fachbereich Mathematik, Technische Hochschule Darmstadt, Schlossgartenstraße 7, D-16100 Darmstadt, Fed. Rep. Germany

Sidney A. MORRIS
Department of Mathematics, La Trobe University, Bundoora, Victoria 3083, Australia

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As the second step in understanding the structure of free compact groups, the center Z of a free compact group FX on a compact space X is here discussed. In particular, we shall show that Z is contained in the identity component $(FX)_0$ and that Z is related to the free compact abelian group F_aX on X, which group we studied extensively in the first of this series of articles. Indeed, if $F'X$ denotes the closure of the commutator subgroup of FX, it is shown here that the function $z \mapsto z(FX): Z_0 \to F_aX = FX/F'X$ is a projective cover (in a sense previously specified); its kernel $Z_0 \cap F'X$ has the character group $\mathbf{Q}/\mathbf{Z} \otimes H^1(X, \mathbf{Z})$. If X is connected and $H^1(X, \mathbf{Z})$ is divisible, then the free compact group is the direct product of its commutator subgroup and the free compact abelian group.

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0. Introduction and overview

This is the second paper in a sequence which exposes the structure of free compact groups. (For an overview of our study of free compact groups, see [3].) In [5] we completely described the structure of free compact abelian groups. For example, we showed that if X is an infinite compact connected metric space, then the dual group of the free compact abelian group on X, F_aX, is topologically isomorphic to the direct sum $H^1(X, \mathbf{Z}) \oplus G$, where G is a rational vector space of dimension $2^{\aleph_0}$ and $H^1(X, \mathbf{Z})$ denotes the first Alexander-Spanier cohomology group. We now describe the structure of the center, ZFX, of the free compact group, FX on X, and determine how far it is from the free compact abelian group. We find, for example, that if X is an infinite compact connected space such that $H^1(X, Z) = \{0\}$, then

$ZFX = F_aX$ and FX is topologically isomorphic to $F_aX \times F'X$, where $F'X$ denotes the commutator subgroup of FX.

For the sake of easy reference, we record the relevant definitions. A *free compact group over a topological space* X with base point x_0 is a compact topological group, FX, together with a continuous function $e_X : X \to FX$ mapping x_0 to the identity 1 of FX such that the following universal property is satisfied: For every base point preserving continuous function $f: X \to G$ into a compact topological group G (with 1 as its base point) there is a unique continuous group morphism $f' : FX \to G$ satisfying $f = f'e_X$. (All topological spaces and topological groups we consider are assumed to be Hausdorff.) If $\mathbf{TOP_0}$ denotes the category of topological spaces and base point preserving maps, and if $\mathbf{KG}$ is the category of compact groups, then $F: \mathbf{TOP_0} \to \mathbf{KG}$ is the left adjoint of the grounding functor $|\ | : \mathbf{KG} \to \mathbf{TOP_0}$ associating with a compact group G its underlying space $|G|$ with base point 1.

Our primary interest will be with connected spaces X; under these circumstances FX will be connected. Some of our results do pertain to the general situation.

To bring our strategy into focus we now proceed to some category theoretical remarks and then general observations on the structure of general compact connected groups.

Several adjoint situations are nearby. Firstly if $\mathbf{COMP_0}$ denotes the category of compact spaces with base points and base point preserving continuous functions, then the inclusion function $\mathbf{COMP_0} \to \mathbf{TOP_0}$ has a left adjoint $\beta : \mathbf{TOP_0} \to \mathbf{COMP_0}$, the Stone-Čech compactification. This functor is a retraction, and $F: \mathbf{TOP_0} \to \mathbf{KG}$ factors through β. In fact, if $b_X : X \to \beta X$ is the unit of this adjunction, then $e_X = e_{\beta X} b_X$ and $e_{\beta X}$ is a homeomorphic embedding. (See [5, Remark 1.4.2], where this is proved for F_a in place of F, the proof is verbatim the same. Also see [3, Proposition 14]. Therefore, without loss of generality, we can and shall restrict our attention to pointed compact spaces X. Further, we shall regard X as a subspace of FX.

If G' denotes the closure of the commutator subgroup of the compact group, G, then the functor $G \mapsto G/G'$ is left adjoint to the inclusion functor $\mathbf{KAB} \to \mathbf{KG}$ of the category of abelian groups into the category of compact groups. The group $FX/(FX)'$ is called the *free compact abelian group* and is denoted F_aX. Our paper [5] was devoted to a complete description of F_aX, we can and shall, therefore, consider the strucuture of F_aX as known. We are consequently faced with the problem of determining the group $(FX)'$, the closure of the commutator subgroup, and an extension problem

$$0 \to (FX)' \to FX \to F_aX \to 0. \tag{1}$$

If G_0 denotes the identity component of the compact group, G, then $G \mapsto G/G_0$ is left adjoint to the inclusion functor $\mathbf{KZG} \to \mathbf{KG}$ of the category of compact 0-dimensional groups into the category of compact groups. The group $FX/(FX)_0$ is called the *free compact zero-dimensional group* or the *free profinite group* and is denoted F_zX. For free profinite groups there is a rather detailed theory due to

Mel'nikov [7]. If X is connected, then $F_zX=\{1\}$. We shall show in our preliminary results that the exact sequence

$$0\to(FX)_0\to FX\to F_zX\to 0 \tag{2}$$

splits in **KG**; that is, FX is a semidirect product $(FX)_0\times_s F_zX$. Here the extension problem is comparatively simple, but we are still left with the determination of $(FX)_0$.

This brings us to the question of the structure of compact connected groups. For easy reference, we collect some background information on compact groups in the following theorem.

0.1. Theorem. *Let G be a compact connected group and $Z=Z(G)$ its center, $Z_0=Z_0(G)$ the identity component of its center. Then we have the following conclusions.*

(a) *The algebraic commutator subgroup of G is closed.*

(b) *$G=Z_0G'$. Specifically, there is an exact sequence of compact groups*

$$0\to Z_0\cap G'\xrightarrow{\alpha} Z_0\times G'\xrightarrow{\beta} G\to 0 \tag{3}$$

with $\alpha(z)=(z^{-1},z)$, $\beta(z,k)=zk$, and with a zero-dimensional group $Z_0\cap G'$.

(c) *There is a closed abelian subgroup A in G such that $G=G'A$ and $G'\cap A=\{1\}$. Specifically, there is an isomorphism of compact groups*

$$\mu: G'\times_s A\to G,$$

$\mu(g,a)=ga$, where $G'\times_s A$ denotes the semidirect product with respect to the action of A on G' by inner automorphisms. In particular, $A=G/G'$.

For a proof of (a), (b), and (c) see [1] and for a proof of (d) see [2].

From the exact sequence (3) we immediately derive an exact sequence of compact abelian groups

$$0\to Z_0\cap G'\xrightarrow{\text{incl}} Z_0\xrightarrow{\beta'} G/G'\to 0 \tag{4}$$

with $\beta'(z)=zG'$. Thus putting $G=FX$, the free compact group on the pointed space X, we obtain a close connection between the free compact abelian group F_aX (which we know very well) and the identity component Z_0FX (which we have yet to determine), namely, the connection expressed by the exact sequence (in which we write $F'X$ for $(FX)'$

$$0\to Z_0FX\cap F'X\to Z_0FX\to F_aX\to 0. \tag{5}$$

It is, therefore, of basic importance to know Z_0FX and $Z_0FX\cap F'X$. Of course, the structure of $F'X$ remains to be determined according to Theorem 0.1(b). We shall address this question in a later paper in order to be able to concentrate here on the center of ZFX.

In order to analyze exact sequences like (4) and (5) we have to recall from [5] the concept of a projective cover of a compact abelian group, and the concept of a characteristic sequence of G. If $G^\wedge$ denotes the character group of G and $d_G: G^\wedge \to \mathbf{Q} \otimes G^\wedge$ the natural map given by $d_G(p) = 1 \otimes p$, then $\ker d_g = \operatorname{tor} G^\wedge$ (the torsion subgroup of $G^\wedge$) and $\operatorname{coker} d_G$ is a torsion group. The *projective cover* $p_g: PG \to G$ of G is the dual of d_G, sometimes the group $PG = (\mathbf{Q} \otimes G^\wedge)^\wedge$ is called the *projective cover of* G. If we set $\Delta G = (\operatorname{coker} d_G)^\wedge$ and note that $G/G_0 \cong (\ker d_G)^\wedge$ we obtain the *characteristic sequence of* G

$$0 \to \Delta G \to PG \xrightarrow{d_G} G \to G/G_0 \to 0. \tag{6}$$

Characteristic sequences are discussed in [5, Chapter 2, § 2.2.] In particular, the characteristic sequence of a free compact abelian group $F_a X$ is well understood (see [5, Theorem 2.2.4 and Corollaries 2.2.5 and 2.2.6]). From there we have $PF_a X \cong C(X, \mathbb{R})^\wedge$ naturally, and for a connected compact space X with at least two points, the characteristic sequence of $F_a X$ is

$$0 \to (\mathbf{Q}/\mathbf{Z} \otimes H^1(X, \mathbf{Z}))^\wedge \to PF_a X \to F_a X \to 0. \tag{7}$$

One of the main results of this paper will be that (5) is the characteristic sequence of $F_a X$; that is, it is equivalent to (7). One immediate consequence of this will be:

0.2. Proposition. *Let X be a compact connected space with a least two points and suppose that* $(\mathbf{Q}/\mathbf{Z} \otimes H^1(X, \mathbf{Z}))^\wedge = \{0\}$. *Then*

$$FX \cong F_a X \times F'X.$$

We make the following observation. For any abelian group A, we consider the exact sequence

$$0 \to \operatorname{div} A \to A \to A_{\text{red}} \to 0$$

where $\operatorname{div} A$ denotes the maximal divisible subgroup of A and where A_{red} is defined by the exact sequence. This sequence induces an isomorphism

$$\mathbf{Q}/\mathbf{Z} \otimes A \to \mathbf{Q}/\mathbf{Z} \otimes A_{\text{red}}.$$

Hence

$$\mathbf{Q}/\mathbf{Z} \otimes H^1(X, \mathbf{Z}) \cong \mathbf{Q}/\mathbf{Z} \otimes H^1(X, \mathbf{Z})_{\text{red}}.$$

Hence the divisibility of $H^1(X, \mathbf{Z})$ suffices for the conclusion of Proposition 0.2.

However, we shall prove more. Indeed, we shall elucidate the structure of $Z_0 FX =$ the identity component of the center of ZFX for arbitrary spaces X. For this purpose, we prove a theorem on the structure of general compact groups which, as far as we know, is new and is of independent interest. This theorem is a generalization of the well-known fact that a compact connected group is nearly the direct product of the identity component of its center and its commutator subgroup; namely, Theorem 0.1(b).

0.3. Theorem. *Let G be a compact group, Z its center, and G' the closure of its commutator subgroup. Then we have the following conclusions*:

(a) *$Z \cap G'$ is zero-dimensional (hence so is $Z_0 \cap G'$).*

(b) $G_0 = (ZG')_0 = Z_0(G')_0$.

0.4. Remark. Let G be a compact group as in Theorem 0.3. Set $G_a = G/G'$ and $G_z = G/G_0$. Also set $G_{az} = (G_a)_z$. Then

(i) $(G_a)_0 = G'G_0/G'$,

(ii) $(G_z)' = G'G_0/G_0$,

(iii) $(G_a)_z \cong (G_z)_a \cong G/G'G_0$, naturally.

Proof. The conditions (i) and (ii) follow from the fact that within the category of compact groups surjective morphisms map components onto components and closures of commutator subgroups onto closures of commutator subgroups. Then $(G_a)_z = G_a/(G_a)_0 = (G/G')/(G'G_0/G') \cong G/G'G_0 \cong (G/G_0)/(G'_0/G_0)$ $= G_z/(G_z)'$. □

From Theorem 0.3 and Remark 0.4 we shall easily deduce the following:

0.5. Corollary. *Under the conditions of Theorem* 0.3

(a) $(G/G')_0 \cong G_0G'/G' = Z_0G'/G' \cong Z_0/(Z_0 \cap G')$.

(b) *The morphism $\zeta = (z \mapsto zG') : Z_0 \to G/G'$ is embedded into an exact sequence*

$$0 \to Z_0 \cap G' \xrightarrow{\text{incl}} Z_0 \xrightarrow{\zeta} G_a \xrightarrow{\text{quot}} G_{az} \to 0. \tag{8}$$

Now we specialize the situation of Theorem 0.3 to the case that G is the free compact group FX over an arbitrary pointed compact space X which does not have two components. We then write Z_0FX for Z_0, further $F'X = G'$ and F_aX for G_a, finally $F_{az}X$ for G_{az}.

The principal result for free compact groups in this paper is the following theorem.

0.6. Theorem. *The exact sequence*

$$0 \to Z_0FZ \cap F'X \xrightarrow{\text{incl}} Z_0FX \xrightarrow{\zeta} F_aX \xrightarrow{\text{quot}} F_{az}X \to 0 \tag{9}$$

is the characteristic sequence of F_aX.

This has a number of immediate consequences in view of our precise knowledge of F_aX.

0.7. Corollary.

(a) *The identity component Z_0FX of the center of the free compact abelian group is the projective cover PF_aX of F_aX.*

(b) *$Z_0FX \cap F'X$ is isomorphic to the character group of $\mathbf{Q}/\mathbf{Z} \otimes H^1(X, \mathbf{Z})$.*

We recall, further, that the character group of $F_{az}X$ is isomorphic to $\mathbf{Q}/\mathbf{Z}\otimes C_{fin}(X, \mathbf{Z})$, where C_{fin} indicates the group of finitely valued locally constant functions.

These results completely clarify the role of the identity component, Z_0FX of the center of FX. Our knowledge of the center itself includes the following information.

0.8. Proposition. *The center ZFX is contained in the identity component F_0X. Moreover, $F_0X \cdot F'X = ZFX \cdot F'X$ and $ZFX \cap F'X$ is zero-dimensional. The torsion subgroup of ZFX is contained in $F'X$.*

We do not know whether this torsion subgroup is in fact dense in $ZFX \cap F'X$. This is, nevertheless, true if X is connected.

1. On the center and the commutator subgroup of a compact group

The purpose of this section is to establish a general structure theorem for compact groups, namely Theorem 0.3 of the Introduction and its Corollary 0.5.

We begin with a structure theorem for compact Lie groups.

1.1. Theorem. *Let G be a compact Lie group, $L(G)$ its Lie algebra, Z its center, and G' the closure of its commutator subgroup. Then $L(G) = L(Z) \oplus L(G')$.*

Proof. Let Ad: $G \to \operatorname{Aut} L(G)$ be the adjoint representation of G on $L(G)$. Then $L(Z)$ is exactly the fixed vector space for Ad (G). Since G is compact, $L(G)$ splits into a direct orthogonal sum of the fixed vector space $L(Z)$ and its orthogonal complement V relative to an Ad (G)-invariant inner product. This inner product is also ad $L(G)$-invariant, where ad: $L(G) \to \operatorname{Der} L(G)$ is the adjoint representation of $L(G)$. Therefore, since $L(Z)$ is an ideal of $L(G)$, the vector space V is also an ideal of $L(G)$.

From the representation theory of compact groups we know that the orthogonal complement V of the fixed vector space is spanned by the elements of the form $\operatorname{Ad}(g)(X) - X$, $g \in G$, $X \in L(G)$. But if we write

$$X * Y = X + Y + \tfrac{1}{2}[X, Y] + \cdots$$

for all sufficiently small X, Y, then we know $\exp(X * Y) = \exp X \exp Y$ as well as $X + Y = \lim n((1/n)X * (1/n)Y)$. Thus we deduce $\operatorname{Ad}(g)(X) - X = \lim n((1/n) \operatorname{Ad}(g)(X) * (1/n)(-X))$; that is,

$$\begin{aligned}\exp t(\operatorname{Ad}(g)X - X) &= \lim\left(\exp \frac{t}{n} \operatorname{Ad}(g)(X) \cdot \exp \frac{t}{n}(-X)\right)^n \\ &= \lim\left(g \exp\left(\frac{t}{n} X\right) g^{-1}\left(\exp\left(\frac{t}{n} X\right)\right)^{-1}\right)^n \in G'\end{aligned}$$

for all $g \in G$, $X \in L(G)$ and all $t \in \mathbb{R}$.

It follows that $\mathrm{Ad}(g)(X)-X \in L(G')$ for all $g \in G$ and all $X \in L(G)$. Hence $V \subseteq L(G')$. If we can show the converse inclusion $L(G') \subseteq V$, we are finished.

We first observe that $L(Z(G_0))$ and $L(G_0')$ are orthogonal and that $Z(G) \subseteq Z(G_0)$. Thus $L(Z(G)) \subseteq L(Z(G_0))$. Hence $L(G_0') \subseteq V$. In order to prove $L(G') \subseteq V$, it suffices to prove this claim for the group G/G_0' in place of G; thus we assume from now on that $G_0' = \{1\}$, that is, that G_0 is abelian. Then $\exp: L(G) \to G$ is a morphism and thus $\exp(\mathrm{Ad}(g)(X)-X) = g(\exp X)g^{-1}(\exp X)^{-1}$. Hence $\exp V = [G, G_0]$ (the group algebraically generated by the commutators $ghg^{-1}h^{-1}$ with $g \in G$ and $h \in G_0$). We claim that $\exp V$ is closed. To see this, let F be the kernel of exp. Since $G_0 = \mathrm{im}\exp$ is a torus, F is a free subgroup generated by elements $e_1, \ldots, e_n$ forming a basis of $L(G)$. Since Z is closed, we can assume that $e_1, \ldots, e_m$ were chosen in such a fashion that they form a basis of $L(Z)$. Then

$$\begin{aligned} V &= \mathrm{span}\{\mathrm{Ad}(g)(e_j)-e_j : j=1,\ldots,n;\ g \in G\} \\ &= \mathrm{span}\{\mathrm{Ad}(g)(e_j)-e_j : j=m+1,\ldots,n;\ g \in G\} \end{aligned}$$

since $\mathrm{Ad}(g)(X) = X$ for $X \in L(Z)$. Now $\mathrm{Ad}(G)$ leaves F invariant, and hence $\mathrm{Ad}(g)(e_j)-e_j \in F$ for $j=1,\ldots,n$. Thus $\mathrm{span}_{\mathbb{Z}}\{\mathrm{Ad}(g)(e_j)-e_j : j=m+1,\ldots,n;\ g \in G\} \subseteq F \cap V$. It follows that $\mathrm{span}(F \cap V) = V$. We conclude that $\exp V = V/(V \cap F)$ is compact, hence closed. Finally, there is a finite subgroup E in G such that $G = G_0E$ (see [6]). Hence $[G, G] = [G_0, E][E, E]$, since G_0 is abelian. Thus $\exp V = [G_0, G] = [G_0, E]$ has finite index in $[G, G]$, and thus $\exp V = [G, G]_0$. Also, this implies that $[G, G]$ is compact, and hence is the closed commutator subgroup G'. Then $V = L(G')$, which is what we had to show. □

1.2. Corollary. *Let G be a compact Lie group, Z its center, and G' the closure of its commutator subgroup. Then*

(i) *$Z \cap G'$ is discrete, and*

(ii) *$G_0 = Z_0G_0'$.*

We now generalize Corollary 1.2 to the case of arbitrary compact groups, and so obtain Theorem 0.3 of the Introduction.

1.3. Theorem. *Let G be a compact group, Z its center, and G' the closure of its commutator subgroup. Then*

(i) *$Z \cap G'$ is zero-dimensional, and*

(ii) *$G_0 = Z_0G_0'$.*

Proof. We know that G is a projective limit of compact Lie groups G/N, where N ranges through a filterbase $\mathcal{N}$ of compact normal subgroups converging to 1. The center $Z(G/N)$ of G/N contains ZN/N and the closed commutator subgroup $(G/N)'$ of G/N is $G'N/N$. Since quotient morphisms of locally compact groups map components onto components, the identity component of $Z \cap G'$ is mapped onto

$$(Z \cap G')_0 N/N = ((Z \cap G')N/N)_0 = (ZN/N \cap G'N/N)_0$$
$$\subseteq (Z(G/N) \cap (G/N)')_0 \quad \text{by the above remarks.}$$

Hence, if the last group is singleton for all N, then $Z \cap G'$ is singleton. Thus, in order to prove (i), it suffices to establish this claim for any compact Lie group, but this was indeed established in Corollary 1.2.

Claim (ii) also has been established for the Lie group case, and hence $(G/N)_0 \subseteq Z(G/N)(G/N)'$, for all N. This means that $(G_0N)/N \subseteq Z(G/N)(G'N)/N$; that is, $G_0 \subseteq Z_N G'$, where Z_N is the complete inverse image of $Z(G/N)$ in G. It then follows that

$$G_0 \subseteq \bigcap \{Z_N G' \colon N \in \mathcal{N}\}$$
$$= (\bigcap \{Z_N \colon N \in \mathcal{N}\})G'$$

(by the compactness of G', and since $\{Z_N \colon N \in N\}$ is a filter base)

$$= ZG', \quad \text{since } Z = \bigcap \{Z_N \colon N \in \mathcal{N}\}.$$

This shows that $G_0 \subseteq ZG'$. Then clearly $G_0 \subseteq (ZG')_0 = Z_0 G'_0 \subseteq G_0$, and so $G_0 = Z_0 G'_0$, as asserted. □

Remark. If we specialize Theorem 1.3 to the case of a connected compact group, G, we recover a classical result (see Theorem 0.1.(c)). If G is not connected, this result still applies to G_0 and yields:

$$G_0 = Z_0(G_0)(G_0)' \tag{$*$}$$

with $Z_0(G_0)$ denoting the identity component of the center of G_0. On the other hand, Theorem 1.3 gives

$$G_0 = Z_0(G)(G')_0. \tag{$**$}$$

Observe that $Z_0(G) \subseteq Z_0(G_0)$, while $(G_0)' \subseteq (G')_0$. So $(**)$ cannot be deduced from $(*)$. Indeed, it is quite possible that G_0 is abelian, while $G_0 = (G')_0$ and $Z_0(G) = \{0\}$, whereas $Z_0(G_0) = G_0$ and $(G_0)' = \{0\}$. Similar remarks apply to the zero-dimensionality of $Z \cap G'$.

As a corollary to this theorem we deduce the following result (Corollary 0.5 of the Introduction) which we shall exploit in our investigation of the free compact group, where the exact sequence obtained will be the characteristic sequence of the free compact abelian group.

1.4. Corollary. *Let G be a compact group, Z its center, and G' the closure of its commutator subgroup. Then the groups $Z \cap G'$ and G/ZG' are both zero dimensional, and there are exact sequences of compact abelian groups*

$$0 \to Z_0 \cap G' \to Z_0 \to G/G' = G_a \to G_a/(G_a)_0 \to 0, \tag{11}$$
$$0 \to Z \cap G' \to Z \to G_a \to G/ZG' \to 0, \tag{12}$$

where the only nonobvious maps are given as follows: $\zeta: Z_0 \to G_a$ and $\zeta: Z \to G_a$ *are given by* $\zeta(z) = zG'$, *and* $G_a \to G/ZG'$ by $gG' \mapsto gZG'$.

Proof. By Theorem 1.3, the group $Z \cap G'$ is zero-dimensional and $G_0 \subseteq ZG'$, whence G/ZG' is zero-dimensional. We note that

$$(G_a)_0 = (G/G')_0 = G_0G'/G' = Z_0G'/G' \text{ (by Theorem 1.3(ii))}$$
$$\cong Z_0/(Z_0 \cap G').$$

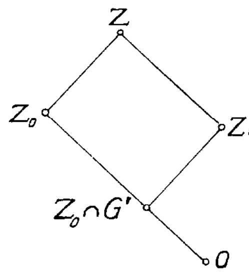
This shows, in particular, exactness of (11) at G_a. Exactness at the other places of (11) is obvious. Exactness of (12) is clear. □

Remark. Roughly speaking, the identity component Z_0 of the center of any compact group, G, differs from the abelianization $G/G' = G_a$ of G just by groups of dimension zero. In Section 2, we shall encounter the case in which the center is contained in the identity component.

1.5. Corollary. *Let* G *be a compact group and assume that* $Z \subseteq G_0$. *Then we have the following conclusions*:

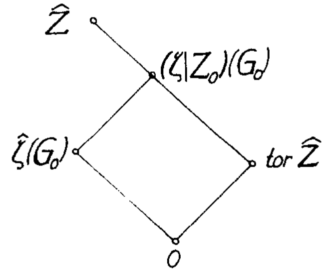
(i) $ZG' = G_0G'$; *that is*, $G_{az} = G/ZG'$.

(ii) $Z = Z_0(Z \cap G')$, $Z_0 \cap G' = Z_0 \cap (Z \cap G')$, *that is, we have a lattice diagram*



In particular, $Z_z = (Z \cap G')/(Z_0 \cap G')$.

(iii) *The dual diagram is*



with $\zeta: Z \to G_a$, $\zeta(z) = zG'$.

Proof. (i) By Theorem 1.3, we have $G_0 \subseteq ZG'$; by hypothesis, $ZG' \subseteq G_0G'$, so $G_0G' = ZG'$. By Corollary 0.5, this implies $G_{az} = G/ZG'$.

(ii) Clearly, $Z_0(Z \cap G') \subseteq Z$. Now let $z \in Z$. Then $z \in G_0 \subseteq Z_0G'$ by 1.3. Hence, $z = z_0g$ with $z_0 \in Z_0$, $g \in G'$, whence $g = zz_0^{-1} \in Z \cap G'$ and thus $z \in Z_0(Z \cap G')$. Trivially, $Z_0 \cap G = Z_0 \cap (Z \cap G)$.

(iii) This follows by duality from the exactness of the sequences (11) and (12) in Corollary 1.4. □

We observe that the exact sequence (11) of Corollary 1.4 is a good candidate for the characteristic sequence of G_a.

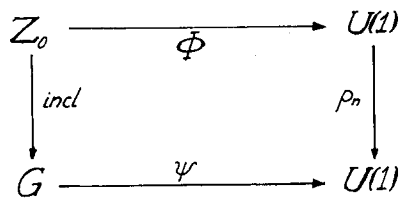
1.6. Lemma. *With the notation of Proposition* 1.4, *the following conditions are equivalent*:

(I) *The exact sequence* (11) *is the characteristic sequence of* G_a.

(II) *The group* $Z_0^\wedge$ *is divisible.*

(III) *For each* $\chi \in \operatorname{im} \zeta^\wedge \subseteq Z_0^\wedge$ *and each natural number, n, the element has an* n*th root in* $Z_0^\wedge$.

(IV) *For each continuous morphism* $\psi : G \to U(1)$ *and each natural number, n, there is a character* $\phi : Z_0 \to U(1)$ *such that the diagram*



commutes with $p_n(z) = z^n$.

Proof. In order to see the equivalence of (I) and (II) we observe that the exact sequence (11) is the characteristic sequence of G_a if and only if Z_0 is projective in the category of compact abelian groups, and this holds precisely when its character group $Z_0^\wedge$ is divisible or, equivalently, Z_0 itself is torsion free. (See [5, Remark 2.1.2].) Trivially, (II) implies (III). We now show the converse implication: The exact sequence

$$0 \to Z_0 \cap G' \to Z_0 \to G_a$$

has the following exact sequence as dual:

$$0 \leftarrow Z_0 \cap G'^\wedge \leftarrow Z_0^\wedge \leftarrow G_a^\wedge. \tag{12}$$

By Theorem 1.3, the group $Z_0 \cap G'$ is zero-dimensional. Hence its character group $Z_0 \cap G'^\wedge$ is a torsion group. From this and the exactness of sequence (12) it follows that every element of Z_0 has some power in $\operatorname{im} \zeta^\wedge$. In order to prove (II), we let $\chi \in Z_0^\wedge$ and let n be a natural number. Then there is a number m such that $\chi^m \in \operatorname{im} \zeta^\wedge$. By condition (III), there is a $\phi \in Z_0$ such that $\phi^{mn} = \chi^m$. Since Z_0 is torsion free, $\phi^n = \chi$ follows.

Now we show that the conditions (III) and (IV) are equivalent. If $q: G \to G_a = G/G_a$ denotes the quotient map, then every $\psi: G \to U(1)$ factors uniquely through q, since $U(1)$ is abelian. Because $q \circ \text{incl} = \zeta$ with $\text{incl}: Z_0 \to G$, condition (IV) is equivalent to saying that for every character $\chi: G_a \to U(1)$ and every natural number n there is a character $\phi \in Z_0$ such that $\chi \circ \zeta = \phi^n$. But this statement is exactly condition (III). □

1.7. Lemma. *The conditions* (I) *through* (IV) *of Lemma* 1.6 *are implied by the following condition*

(III*) *If* $\zeta: Z \to G_a$ *is given by* $\zeta(g) = gG'$, *then for every element* $\chi \in \text{im}\,\chi^\wedge \subseteq Z^\wedge$ *and every natural number n then element* χ *has an nth root in* $Z^\wedge$.

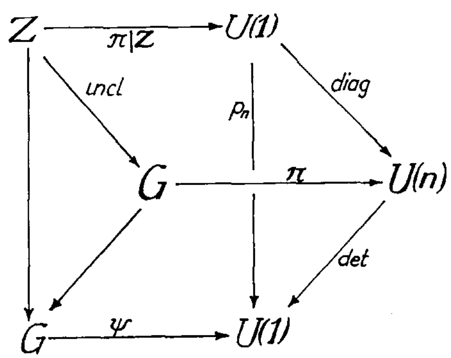
This condition in turn is implied by

(V) *For each morphism* $\psi: G \to U(1)$ *and each natural number n, there is an irreducible representation* $\pi: G \to U(n)$ *such that* $\psi(g) = \det \pi(g)$ *for all* $g \in G$.

Proof. Trivially, (III*) implies (III). Now we show that (V) implies (IV*), where condition (IV*) arises from condition (IV) simply by replacing Z_0 by Z throughout. Since the equivalence of (III*) and (IV*) follows exactly as in Lemma 1.4, this will prove our assertion. Suppose that a morphism $\psi: G \to U(1)$ and a natural number n are given. Let $\pi: G \to U(n)$ be the irreducible representation according to (V). If $g \in Z$, then by the irreducibility of π and Schur's Lemma, $\pi(g)$ is of the form $\phi(g)E_n$ with the $n \times n$ identity matrix E_n and a complex number $\phi(g)$. Now $\phi: Z \to U(1)$ is a character, and for $g \in Z$ we have $\det \pi(g) = \phi(g)^n$, as asserted in (IV). □

In the next section we shall show that the lifting condition (V) is indeed satisfied by free compact groups.

The situation of condition (V) is illustrated in the following diagram:



2. The center of a free compact group

In order to prove Theorem 0.6 of the Introduction we now verify condition (V) of Lemma 1.7 for the free compact group $G = FX$.

2.1. Lemma. *Let $G = FX$ be the free compact group on a compact space X with a base point and at least two further distinct points a and b. Then condition* (V) *of Lemma* 1.7 *is satisfied.*

Proof. Let $s: U(1) \to U(n)$ be defined by $s(z) = \text{diag}(z, 1, \ldots, 1)$. Thus

(i) $\det s(z) = z$.

The group $SU(n)$ is topologically generated by two elements g and h (see for example [8, Proposition 4]). Since $SU(n)$ is path-connected there is a continuous function $f_0: X \to SU(n)$ such that

(ii) $f_0(a) = gs(\chi(a))^{-1}$, $f_0(b) = hs(\chi(b))^{-1}$.

Now let $f(x) = f_0(x)s(\chi(x))$. Then

(iii) $\det f(x) = \chi(x)$ by (i), and

(iv) $f(a) = g$, $f(b) = h$ by (ii).

By the universal property of F, there is a representation $\pi: FX \to U(x)$ with

(v) $\pi|X = f$.

Then $\pi\,(FX)$ contains $SU(n)$, since $\{f(a), f(b)\}$ topologically generates $SU(n)$. Hence π is irreducible. By (iii) and (iv), however, $\det \circ \pi$ and χ agree on X, hence $\psi(g) = \det \pi(g)$ for all $g \in FX$, which was to be shown. □

This information gives us Theorem 0.6 of the Introduction, the major result of this paper for free compact groups.

2.2. Theorem. *For any compact space X the following sequence is the characteristic sequence of the free compact abelian group, F_aX:*

$$0 \to Z_0(FX) \cap F'X \to Z_0\,(FX) \xrightarrow{\zeta} F_aZ \to F_{az} \to 0, \tag{13}$$

where $Z_0(FX)$ is the identity component of the center of the free compact group, FX, where $F'X$ denotes the closure of the commutator subgroup of FX, where $F_aX = FX/F'X$ and *$F_{az}X = F_aX/(F_aX)_0$,* and *where $\zeta(g) = gF'X$.*

Proof. If X has at least three points, then by Lemmas 2.1 and 1.6 and Corollary 1.5 the proof is completed. If X has one point, then all groups in sight are singleton, and the assertion is true by default. There remains the case of two points. Then FX is abelian and thus agrees with the free compact abelian group F_aX. In this case, $F'X = \{0\}$ and $Z = FX$, whence $Z_0 = (F_aX)_0$, and the sequence in question becomes

$$0 \to (F_aX)_0 \to F_aX \to F_{az}X \to 0$$

which is the characteristic sequence of F_aX by [5, 2.2.4 (ii)]. □

In [5] we have a complete structure theory of F_aX and all terms of the characteristic sequence of F_aX. Now we have obtained the characteristic sequence (13) of F_aX

starting from the free compact group FX. This allows us now to clarify the structure of all terms of (13).

2.3. Theorem. *The character groups of the terms of the sequence* (13) *are given by*
(i) $(Z_0(FX)\cap F'X)^\wedge = \mathbf{Q}/\mathbf{Z}\otimes H^1(X,\mathbf{Z}) = \mathbf{Q}/\mathbf{Z}\otimes H^1(X,\mathbf{Z})_{\mathrm{red}}$.
(ii) $Z_0(FX)^\wedge = C(X,\mathbf{Q}) \cong (\mathbf{Q}^\wedge)^{\max(2^{\aleph_0},\omega(X)^{\aleph_0})}$.

Proof. For Part (i), see [5, Theorem 2.2.4], for Part (ii) see [5, Theorem 2.2.4, Remark 2.2.5, and Theorem 1.5.4(ii)]. □

Remark. The structure of $F_{\mathrm{a}}X$ and $F_{\mathrm{az}}X$ are precisely described in [5], notably in Theorem 1.5.4, and Corollary 1.5.3(ii), respectively. For instance $(F_{\mathrm{az}}X)^\wedge \cong C_{\mathrm{fin}}(X,\mathbf{Q}/\mathbf{Z})$.

Lemma 2.1 gives more. Indeed let us first record the following fact on the duality of abelian groups:

2.4. Definition and Remark. Let A be a (discrete) abelian group and $G = A^\wedge$ its dual group. For a subgroup B of A, let $B^\perp$ denote its annihilator in G. Let $D(A)$ denote the subgroup of all divisible elements of A; that is, all elements $a \in A$ for which the equation $n\cdot x = a$ is solvable in A for all $n \in \mathbf{Z}$. Then, if $\overline{\mathrm{tor}}(G)$ denotes the closure of the torsion subgroup of G, we have

$$\overline{\mathrm{tor}}(G) = D(A)^\perp.$$

Now we utilize Lemma 2.1 to record the following result:

2.5. Theorem. *If* $\zeta : ZFX \to F_{\mathrm{a}}X$ *is as in Theorem* 2.2, *then*

$$\overline{\mathrm{tor}}(Z) \subseteq Z \cap F'X.$$

Proof. We let $\zeta^\wedge$ denote the dual of ζ. Then a character $\chi : Z \to U(1)$ of Z is in $\zeta^\wedge(F_{\mathrm{a}}X)$ if and only if there is a morphism $\psi : FX \to U(1)$ such that $\chi = \psi | Z$. Condition (V) of Lemma 1.7 clearly states for $G = FX$ that every element of $\zeta^\wedge(F_{\mathrm{a}}X)$ is a divisible element in $Z^\wedge$. Thus

$$\zeta^\wedge(F_{\mathrm{a}}X) \subseteq D(Z^\wedge).$$

But by the exactness of sequence (13) of Theorem 2.2, we have

$$\zeta^\wedge(F_{\mathrm{a}}X) = (Z \cap F'X)^\perp.$$

Consequently, we have

$$(Z \cap F'X)^{\perp} \subseteq D(Z^{\wedge}).$$

By duality, the claim of the theorem follows. □

2.6. Theorem. *If in addition to the hypothesis of Theorem* 2.5, X *is connected then* $\overline{\text{tor}}(Z) = Z \cap F'X$.

Proof. If X is connected, then FX is connected and $F'X$ is semisimple. Then $Z \cap F'X$ is the center of a compact connected semisimple group; that is, there is a surjective morphism $S \to FX$, where S is a direct product of a family of simple simply connected compact Lie groups. Since the center of a compact semisimple Lie group is finite, ZS has a dense torsion subgroup. Consequently, $ZF'X = Z \cap F'X$ has a dense torsion subgroup. Thus $Z \cap F'X \subseteq \overline{\text{tor}}(Z)$. But the converse containment follows from Theorem 2.5. Hence the theorem is proved. □

2.7. Open Question. Does $\overline{\text{tor}}(Z) = Z \cap F'X$ hold for all compact pointed spaces X?

We shall now show that the center, Z, of a free compact group, FX, is contained in the identity component, F_0X, of FX. For this purpose, recall that $F_zX = FX/F_0X$ is the free compact zero-dimensional (that is, profinite) group over the compact pointed space X (see [3]). The following is an observation of a category theoretical nature (see [3, Proposition 1.4(iii)]).

2.8. Proposition. *For a compact space* X, *let* X/conn *be the zero-dimensional space of all components of* X. *Then the quotient map* $\gamma_X : X \to X/\text{conn}$ *induces an isomorphism* $F_z\gamma_X : F_zX \to F(X/\text{conn})$.

Proof. Let $\eta_X : X \to F_zX$ be the front adjunction. Since F_zX is a zero-dimensional space, η_X factors through $\gamma_X : X \to X/\text{conn}$; that is, there is continuous function $\phi : X/\text{conn} \to F_zX$ such that $\eta_X = \phi\gamma_X$. By the universal property of $F_z(X/\text{conn})$ there is a unique morphism $\phi' : F_z(X/\text{conn}) \to F_zX$ such that $\phi'\gamma_{X/\text{conn}} = \phi$. Now $\eta_X = \phi\gamma_X = \phi'\eta_{X/\text{conn}}\gamma_X = \phi'(F\gamma_X)\eta_X$. By the uniqueness in the universal property, this implies $\phi'(F\gamma_X) = \text{id}$, and since $F\gamma_X$ is surjective, this proves the claim. □

The following result appears in [4]. The work of Mel'nikov [7] contains a more general result for a more special class of spaces, namely, one point compactifications of discrete spaces.

2.9. Theorem. *Let X be a compact pointed space with at least three points. Then the center Z of the free compact group FX is contained in the identity component F_0X.*

Proof. First we note that it suffices to show that the center of the free compact zero-dimensional group F_zX is singleton; for ZF_0X/F_0X is contained in the center of F_zX. The proof of this claim is by contradiction. We assume that F_zX contains a central element $z_0 \neq 1$. Then we find a finite quotient $q: FX \to G$ with $z = q(z_0) \neq 0$. If we denote $q(e(X))$ by Y, then Y is a generating set of G. The next step utilizes a familiar lifting technique based on the universal property of F_zX.

Claim 1. Let E be a finite group and $p: E \to G$ a quotient morphism. If $s: Y \to E$ is any function satisfying $ps = \mathrm{id}_Y$, then there is a unique morphism $Q: FX \to E$ with $pQ = q$. The subgroup im Q of E is generated by $s(Y)$ and centralizes $Q(z_0)$.

Proof of Claim 1. By the universal property, the function sqe: $X \to E$ determines a unique morphism $Q: FX \to E$ with $Qe = sqe$. Then $pQe = psqe = qe$, and by the uniqueness aspect of the universal property, $pQ = q$ follows. Since FX is topologically generated by $e(X)$ and since E is discrete, the group im Q is generated by $Q(e(X)) = sqe(X) = s(Y)$. Since z_0 is central in FX, then $z = Q(z_0)$ is central in im Q. This completes the proof of Claim 1.

Our objective is to obtain a contradiction by choosing the parameters E and s. For this purpose we consider a finite-dimensional vector space M over a finite field K and assume that G operates linearly on M on the left. Then M is a G- and a $K(G)$-module, where $K(G)$ is the group ring of G over K. We take for E the semidirect product $E = M \times_s G$; that is, the cartesian product with the multiplication $(m, g)(n, h) = (m + gn, gh)$. The first application results from taking $M = K(G)$ with multiplication on the left as the action.

Claim 2. For each $y \in Y$ there is a natural number a with a $y^a = z$.

Proof of Claim 2. We apply Claim 1 with $E = K(G) \times_s G$, and $p = \mathrm{pr}_2: E \to G$; furthermore we take $s: Y \to E$ to be defined by $s(y) = (1, y)$ for all $y \in Y$. If we write $Q(z_0) = (c, z)$ and recall that (c, z) is centralized by im Q, hence in particular commutes with all elements $(1, y)$, we obtain $(1 + yp, yz) = (1, y)(c, z) = (c, z)(1, y) = (c + z, zy)$, whence $1 + yc = c + z$. This may be written as

$$1 - z = (1 - y)c \quad \text{for all } y \in Y. \tag{14}$$

In the group ring $K(G)$ the element c is of the form $\sum r_g \cdot g$ with $r_g = r(g) \in K$ and the summation extended over all $g \in G$. One observes that the relations (14) then are equivalent to the equations

$$\begin{aligned} &\text{(i)} && r(y^{-1}) = r(1) - 1, \\ &\text{(ii)} && r(y^{-1}z) = r(z) + 1, \\ &\text{(iii)} && r(y^{-1}g) = r(g) \quad \text{for all } g \neq 1, \quad z, y \in Y. \end{aligned} \tag{15}$$

Assume now that the claim is false. Then we would find a $y \in Y$ such that no power of y equals z. We consider the coefficients $r(g)$ for g in the cyclic subgroup

$\{1, y^{-1}, y^{-2}, \ldots, y^{-n+1}\}$, where n is the order of y. From (15)(i) and (iii), we conclude inductively that $r(y^{-m}) = r(1) - 1$ for $m = 1, \ldots, n$ and thus arrive at the contradiction $r(1) = r(y^{-n}) = r(1) - 1$. This proves Claim 2.

In order to conclude with a contradiction, after Claim 2 it suffices to find a finite quotient map $Q: FX \to H$ such that $Q(z_0)$ is not contained in the subgroup generated by some $y \in Q(e(X))$.

Claim 3. The group FX is nonabelian and $q: FX \to G$ is a nonabelian finite quotient with $z = q(z_0) \neq 1$. Apply Claim 1 with $E = K(G) \times_s G$ and $p = \mathrm{pr}_2: E \to G$ and with $s: Y \to E$ defined by $s(y) = (0, y)$ for all $y \in Y$ with the exception of one $u \in Y$ for which we define $s(u) = (u, u)$. Then $H = \mathrm{im}\, Q$ is a finite quotient of FX with the property that for at least one generator $Q(e(X))$ of H the element $Q(z_0)$ is not in the subgroup generated by $Q(e(x))$.

Proof of Claim 3. Since FX is nonabelian, there are finite nonabelian quotients G of the required sort. Moreover, the generating set Y has to contain, outside 1 and u, at least one other element v. Suppose now by way of contradiction, that $Q(z_0) = (c, z)$ equals $s(v)^a = (0, v)^a = (0, v^a)$ for some natural number a. By Claim 1, then $(u, u) = s(u)$ commutes with $(c, z) = (0, v^a)$, whence $(u, uv^a) = (u, u)(0, v^a) = (0, v^a)(u, u) = (v^a u, v^a u)$. This implies $v^a u = u$ and thus $v^a = 1$, which entails $z = 1$ in contradiction with the choice of q. Thus any x with $e(x) = v$ satisfies the conclusion of Claim 3.

By the remark preceding Claim 3, and since X has at least three points and FX is therefore nonabelian, the proof of the Theorem is now complete. □

By Corollary 1.5 we now obtain the following result.

2.10. Theorem. *For any compact pointed space X, the following conclusions hold:*

(i) $(ZFX)(F'X) = (F_0X)(F'X)$; *that is,* $F_{az} = FX/(ZFX)(F'X)$.

(ii) $(ZFX) = (Z_0FX)(ZFX \cap F'X), (Z_0FX) \cap F'X = (Z_0FX) \cap ((ZFX) \cap F'X)$.

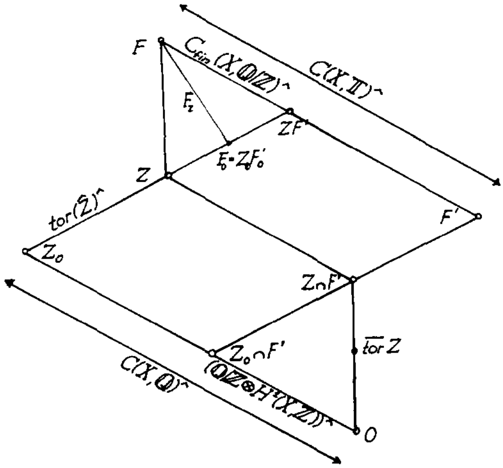
(iii) *F_0X is a direct product of $Z_0(FX)$ and $F_0X \cap F'X$ if and only if $H^1(X, \mathbf{Z})$ is divisible.*

Proof. Statements (i) and (ii) follow from Corollary 1.5. In order to prove (iii) we observe that $H^1(X, \mathbf{Z})$ is divisible if and only if $\mathbf{Q}/\mathbf{Z} \otimes H^1(X, \mathbf{Z}) = \{0\}$ which, by Theorem 2.3(i), means $Z_0(FX) \cap F'X = \{0\}$. The result then follows from (ii) and Theorem 1.3. □

2.11. Corollary. *Let X be a connected compact pointed space such that $H^1(X, \mathbf{Z})$ is divisible. Then the free compact group FX is the direct product of the free compact abelian group F_aX and its commutator subgroup $F'X$.*

For a lattice diagram of the situation of Theorem 2.10, see the lattice diagram in Corollary 1.5.

We summarize all of our results on the center of a free compact group on a compact pointed space X in the following lattice diagram:



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