PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 2, June 1988

SEQUENTIAL CONDITIONS AND FREE PRODUCTS OF TOPOLOGICAL GROUPS

SIDNEY A. MORRIS AND H. B. THOMPSON

(Communicated by Dennis Burke)

ABSTRACT. If A and B are nontrivial topological groups, not both discrete, such that their free product $A \amalg B$ is a sequential space, then it is sequential of order ω_1.

1. Introduction. In [**15**], Ordman and Smith-Thomas prove that if the free topological group on a nondiscrete space is sequential then it is sequential of order ω_1. In particular, this implies that free topological groups are not metrizable or even Fréchet spaces. Our main result is the analogue of this for free products of topological groups. More precisely we prove that if A and B are nontrivial topological groups not both discrete and their free product $A \amalg B$ is sequential, then it is sequential of order ω_1. This result is then extended to some amalgamated free products. Our theorem includes, as a special case, the result of [**10**]. En route we extend the Ordman and Smith-Thomas result to a number of other topologies on a free group including the Graev topology which is the finest locally invariant group topology. (See [**12**].) It should be mentioned, also, that Ordman and Smith-Thomas show that the condition of $A \amalg B$ being sequential is satisfied whenever A and B are sequential k_ω-spaces.

2. Preliminaries. The following definitions and examples are based on Franklin [**3**, **4**] and Engelking [**2**].

DEFINITIONS. A subset U of a topological space X is said to be *sequentially open* if each sequence converging to a point in U is eventually in U. The space X is said to be *sequential* if each sequentially open subset of X is open.

REMARKS. A closed subspace of a sequential space is sequential. A subspace of a sequential space need not be sequential. (See Example 1.8 of [**3**].)

DEFINITIONS. For each subset A of a sequential space X, let $s(A)$ denote the set of all limits of sequences of points of A. The space X is said to be *sequential of order* 1 if $s(A)$ is the closure of A for every A.

The higher sequential orders are defined by induction. Let $s_0(A) = A$, and for each ordinal $\alpha = \beta + 1$, let $s_\alpha(A) = s(s_\beta(A))$. If α is a limit ordinal, let $s_\alpha(A) = \bigcup\{s_\beta(A)\colon \beta < \alpha\}$. The *sequential order* of X is defined to be the least ordinal α such that $s_\alpha(A)$ is the closure of A for every subset A of X.

REMARKS. The sequential order always exists and does not exceed the first uncountable ordinal ω_1. Sequential spaces of order 1 are also known as *Fréchet*

Received by the editors March 14, 1986 and, in revised form, February 6, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 22A05, 54D55, 20E06.
Key words and phrases. Sequential space, sequential order, free product of topological groups, Graev topology, free abelian topological group.

spaces. Clearly any metrizable space is a Fréchet space; however, there exist sequential spaces which are not Fréchet and Fréchet spaces which are not metrizable. Indeed, for each ordinal $\alpha \leq \omega_1$ there exists a sequential space of that order. The key example is due to Arhangel′skiĭ and Franklin [**1**].

By S_1 we mean a space consisting of a single convergent sequence $s_1, s_2, \ldots,$ together with its limit point s_0 taken as the basepoint.

The space S_2 is obtained from S_1 by attaching to each isolated point s_n of S_1 a sequence $s_{n,1}, s_{n,2}, \ldots,$ converging to s_n. Thus S_2 can be viewed as a quotient of a disjoint union of convergent sequences; we give it the quotient topology. Inductively, we obtain the space S_{n+1} from S_n by attaching a convergent sequence to each isolated point of S_n and giving the resultant set the quotient topology.

Let S_ω be the union of the sets $S_1 \subset S_2 \subset S_3 \subset \cdots$, with the weak union topology (a subset of S_ω is closed if and only if its intersection with each S_n is closed in the topology of S_n).

It is shown in [**1**] that each S_n is sequential of order n and S_ω is sequential of order ω_1.

DEFINITION. Let X be a topological space with distinguished point e, and $FA(X)$ a topological group which contains X as a subspace and has e as its identity element. Then $FA(X)$ is said to be the (*Graev*) *free abelian topological group on X* if for any continuous map ϕ of X into any abelian topological group H such that $\phi(e)$ is the identity element of H, there exists a unique continuous homomorphism $\Phi\colon FA(X) \to H$ with $\Phi|H = \phi$.

For a recent survey of free abelian topological groups see [**11**].

If in the above definition the word abelian is everywhere deleted we obtain the definition of the (*Graev*) *free topological group on the space X*.

DEFINITION. Let N be a common subgroup of topological groups A and B. The topological group $A \amalg_N B$ is said to be the *free product of the topological groups A and B with amalgamated subgroup N* if

(i) A and B are topological subgroups of $A \amalg_N B$,

(ii) every pair ϕ_1, ϕ_2 of continuous homormorphism of A and B, respectively into any topological group H which agree on N extend uniquely to a continuous homomorphism of $A \amalg_N B$ into H.

If $N = \{e\}$, the identity element, then $A \amalg_N B = A \amalg B$, the free product of the topological groups A and B.

In order to prove that the free topological group is Hausdorff, Graev [**5**] introduced a specific topology τ_G on the free group $F(X)$ on the set $X \setminus \{e\}$. We refer to this topology as the *Graev topology.* It is the finest locally invariant group topology on $F(X)$ which induces the given topology on X. (A topological group is said to be *locally invariant* if every neighborhood of e contains a neighborhood of e invariant under inner automorphisms.) The topology is described *and analyzed* in [**12**]. In particular, Graev shows τ_G is Hausdorff and so the free topological group is Hausdorff.

Using a much more delicate and technical argument, Graev [**6**] proved that if A and B are any Hausdorff topological groups, then the free product $A \amalg B$ is Hausdorff. (In [**8**], Graev's argument is extended to the case of amalgamated free products, where the subgroup being amalgamated is central.) Once again Graev does this by putting a coarser topology on $A \amalg B$ than the free product topology and showing this coarser topology is Hausdorff.

3. Results.

LEMMA. *Let G be a sequential Hausdorff topological group with identity e and an infinite subspace $Y = \bigcup_{n=1}^{\infty}\{y_n\} \cup \{e\}$ where the sequence y_n converges to e. Let $FA(X)$ be the (Graev) free abelian topological group on a set X with distinguished point e. If there exists a continuous homomorphism $\Phi\colon G \to FA(X)$ such that $\Phi|Y$ is one-to-one, then G has a closed subspace homeomorphic to S_ω.*

PROOF. First we choose a subsequence $\{g_i\}$ of $\{y_i\}$ with the property that, if $g_{i_1}^{\varepsilon_1} \cdots g_{i_r}^{\varepsilon_r} = g_{j_1}^{\eta_1} \cdots g_{j_s}^{\eta_s}$ for some r, s, $\varepsilon_1, \dots, \varepsilon_r$, $\eta_1, \dots, \eta_s \in \mathbf{N}$, the natural numbers, and $\{g_{i_1}, \dots, g_{i_r}, g_{j_1}, \dots, g_{j_s}\} \subseteq \{g_i\}$, then $\sum_k\{\varepsilon_k \colon g_{i_k} = y\} = \sum_p\{\eta_p \colon g_{j_p} = y\}$ for all $y \in Y$; here we adopt the convention that the empty sum is zero. This subsequence can be chosen as follows. Since Y is infinite and $\{y_i\}$ converges to e in the Hausdorff group G, deleting some terms and relabelling if necessary we can assume that $y_i \neq y_j \neq e$, for all i, $j \in \mathbf{N}$ with $i \neq j$. Since $\Phi\colon G \to FA(X)$ is a homormorphism and $\Phi|Y$ is one-to-one, it suffices to choose $\{g_i\}$ such that the sequence $\{\Phi(g_i)\}$ has the required property. Since $\Phi\colon Y \to FA(S)$ is continuous and one-to-one, the set $\{z_i \colon z_i = \Phi(y_i), i \in \mathbf{N}, y_i \in Y\} \cup \{e\}$ is compact in $FA(X)$ and $z_i \neq z_j \neq e$, for all i, $j \in \mathbf{N}$ with $i \neq j$. Let $FA_n(x)$ denote the set of words in $FA(X)$ of length less than or equal to n with respect to X. Thus $\{z_i\} \subseteq FA_N(X)$, for some $N \in \mathbf{N}$, and in reduced form $z_i = x_{i,1}^{\delta_{i,1}} \cdots x_{i,l(i)}^{\delta_{i,l(i)}}$, for some $l(i) \in \mathbf{N}$, $0 \neq \delta_{i,j} \in \mathbf{Z}$, where $\sum_{j=1}^{l(i)} |\delta_{i,j}| \leq N$, each $x_{i,s} \in X$, and $x_{i,j} \neq x_{i,k}$ for $j \neq k$. (It is easily proved and certainly in the folklore that any compact subset of $FA(X)$ is contained in $FA_N(X)$ for some N. This can be proved most easily by observing that it is true when X is compact, since $FA(X)$ is then a k_ω-space with k_ω-decomposition $FA(X) = \bigcup FA_n(X)$. The noncompact case can then be established using Stone-Čech compactification in the manner described in [**7**].)

We shall show how to choose a subsequence Z of the z's such that each z in this subsequence has an x in its reduced representation which does not occur in the reduced representation of any other z in the subsequence. This will be done in two steps.

First, we choose a subsequence of the z's such that each z in this subsequence has an x in its reduced representation which does not occur in the reduced representation of any z further along the subsequence. If $T \subseteq \mathbf{N}$ we set $S(T) = \{x_{i,j} \colon i \in T, 1 \leq j \leq l(i)\}$. Let $T \subseteq \mathbf{N}$ be an infinite subset. Since $z_i \neq z_j$ for $i \neq j$, $\{z_i \colon i \in T\}$ is infinite, and the number of words of length $\leq N$ on a finite subset of X is finite, $S(T)$ is infinite and there is $x_{p,q} \in S(T)$ and an associated infinite subset $\overline{T} \subseteq \{i \colon i \in T, i > p\}$ such that $x_{p,q} \notin S(\overline{T})$. Let $T_0 = \mathbf{N}$, and choose $x_{i_1,j_1} \in S(T_0)$ and associated infinite subset $T_1 \subseteq \{i \colon i \in T_0, i > i_1\}$ such that $x_{i_1,j_1} \notin S(T_1)$. Having chosen x_{i_n,j_n} and T_n we choose $x_{i_{n+1},j_{n+1}}$ and associated infinite subset $T_{n+1} \subseteq \{i \colon i \in T_n, i > i_{n+1}\}$ such that $x_{i_{n+1},j_{n+1}} \in S(T_n) \backslash S(T_{n+1})$. We note that $i_{n+1} \in T_n \setminus T_{n+1}$ for each n. This completes the first step.

The sequence Z we seek will be a subsequence of that chosen in the previous paragraph. Let K, $U \subseteq \{i_n \colon n \in \mathbf{N}\}$ be finite and infinite subsets, respectively. As $S(K)$ is finite, U is infinite, and $x_{i_p,j_p} \neq x_{i_m,j_m}$, for all $p \neq m$, there is an infinite subset $\overline{U} \subseteq \{i_p\colon i_p \in U, i_p > k$ for all $k \in K\}$ such that $x_{i_p,j_p} \notin S(K)$, for all $i_p \in \overline{U}$. Let $K_1 = \{i_1\}$, $U_1 = \{i_p\colon p \in \mathbf{N}, i_p > k$ for all $k \in K_1\}$ and set $x_{i_{n_1},j_{n_1}} = x_{i_1,j_1}$. Having chosen $x_{i_{n_m},j_{n_m}}$, K_m, and U_m let $i_{n_{m+1}}$ be the first element of $U_{m'}K_{m+1} = K_m \cup \{i_{n_{m+1}}\}$ and choose the associated infinite subset $U_{m+1} \subseteq \{i_P\colon i_p \in U_m, i_p > k$ for all $k \in K_{m+1}\}$ such that $x_{i_p,j_p} \notin S(K_{m+1})$ for all $i_p \in U_{m+1}$. This gives the required sequence $Z = \{z_{i_{n_m}}\}$.

Now set $g_m = y_{i_{n_m}}$. By deleting terms and relabelling the $x_{i,j}$'s, $\delta_{i,j}$'s and z_i's, if necessary, can assume that $i_{n_m} = m$ and $j_{n_m} = 1$, for all $m \in \mathbf{N}$ so that $z_m = \Phi(g_m)$ and $z_m = x_{m,1}^{\delta_{m,1}} \cdots x_{m,l(m)}^{\delta_{m,l(m)}}$ in reduced form, where $x_{m,1}$ does not occur in the reduced representation of z_k, for all $k \neq m$. Suppose now $z_{i_1}^{\varepsilon_1} \cdots z_{i_r}^{\varepsilon_r} = z_{j_1}^{\eta_1} \cdots z_{j_s}^{\eta_s}$. Since $x_{m,1}$ occurs only in the reduced form of z_m and $\varepsilon_1, \ldots, \varepsilon_r, \eta_1, \ldots, \eta_s \in \mathbf{N}$ then $\sum_k \{\varepsilon_k\colon z_{i_k} = z_m\} = \sum_p \{\eta_p\colon z_{j_p} = z_m\}$, as required. Note that $\sum_{i=1}^r \varepsilon_i = \sum_{i=1}^s \eta_i$ in this case. Set $Y' = \bigcup_{i=1}^\infty \{g_i\} \in \{e\}$.

We shall produce a closed embedding $\theta\colon S_\omega \to G$ analogous to that of Ordman and Smith-Thomas [**15**, Theorem 3.7]. As in Ordman and Smith-Thomas enumerate the sequences of S_ω as follows. Denote the single sequence of S_1 by $t_1 = s_1, s_2, s_3, \ldots$. Denote the sequence of S_2 converging to s_1 by $t_2 = s_{1,1}, s_{1,2}, s_{1,3}, \ldots$. Use a diagonalization process to enumerate all sequences of S_ω. The limits of the sequences $t_3, t_4, t_5, t_6, t_7, t_8, \ldots$ are respectively $s_{1,1}, s_2, s_{1,1,1}, s_{1,2}, s_{2,1}, s_3, \ldots$. The function θ is constructed to map each sequence t_i to the set of "words" of length precisely i with respect to $Y' \setminus \{e\}$ in G: that is, elements of G which can be expressed as a product of exactly i members of $Y' \setminus \{e\}$ counting multiplicities. Let $\theta(s_0) = e$, $\theta(s_i) = g_i$,

$$\theta(s_{i_1,\ldots,i_n,m}) = \theta(s_{i_1,\ldots,i_n}) g_m^j,$$

where $j = l-$length in G with respect to $Y' \setminus \{e\}$ of $\theta(s_{i_1,i_n})$ and l is such that the sequence t_l converges to $s_{i_1,\ldots,i_n}$. Then

$$\begin{aligned}
\theta(t_2) &= g_1g_1, g_1g_2, g_1g_3, \ldots,\\
\theta(t_3) &= g_1^2g_1, g_1^2g_2, g_1^2g_3, \ldots,\\
\theta(t_4) &= g_2g_1^3, g_2g_2^3, g_2g_3^3, \ldots,\\
\theta(t_5) &= g_1^3g_1^2, g_1^3g_2^2, g_1^3g_3^2, \ldots,\\
\theta(t_6) &= g_1g_2g_1^4, g_1g_2g_2^4, g_1g_2g_3^4, \ldots.
\end{aligned}$$

Clearly θ is continuous. We show θ is one-to-one. Suppose

$$\theta(s_{i_1,\ldots,i_r}) = g_{i_1}^{\varepsilon_1} \cdots g_{i_r}^{\varepsilon_r} = g_{j_1}^{\eta_1} \cdots g_{j_s}^{\eta_s} = \theta(s_{j_1,\ldots,j_s})$$

for some $r, s, \varepsilon_1, \ldots, \varepsilon_r, \eta_1, \ldots, \eta_s \in \mathbf{N}$ and $g_{i_1}, \ldots, g_{i_r}, g_{j_1}, \ldots, g_{j_s} \in \{g_i\}$. By the first part of the proof $\sum_{k=1}^r \varepsilon_k = \sum_{k=1}^s \eta_k = l$, say, so that $s_{i_1,\ldots,i_r} = s_{k_1,\ldots,k_v,m} \in t_l$ and $s_{j_1,\ldots,j_s} = s_{k_1,\ldots,k_v,n} \in t_l$, some $k_1, \ldots, k_v$, $m, n \in \mathbf{N}$. By the first part of the proof $m = n$ and thus θ is one-to-one. Now set $f = \Phi \circ \theta\colon S_w \to FA(X)$. Noting that $\Phi|Y$ is one-to-one the argument in [**15**], which carries over directly to the abelian case, shows that f is a closed embedding. Therefore, θ is an embedding of S_ω into G. We now show θ is a closed mapping. Let $\omega \in \overline{\theta(S_\omega)}$; then there exists a sequence $w_n \in \theta(S_\omega)$ such that w_n converges to w. Thus $\Phi(w_n)$ converges to $\Phi(w)$. As $\bigcup_{n=1}^\infty \{\Phi(w_n)\} \cup \{\Phi(w)\}$ is compact in $FA(X)$ it is contained in $FA_N(X)$ for some N. Let $u_n \in S_w$ satisfy $\theta(u_n) = w_n$. As $f(u_n) = \Phi(w_n)$ and $f(S_w) \cap FA_N(X)$

consists of N convergent sequences there exists u_{n_i} of u_n converging to $u \in S_w$ such that $w_{n_i} = \theta(u_{n_i})$ converges to $\theta(u) = w \in \theta(S_w)$. Thus $\theta(S_w)$ is closed and θ is a closed embedding. □

The following proposition is an immediate consequence of the Lemma and Proposition 1 of [**12**] and is of interest in its own right. (Proposition 1 of [**12**] yields that the quotient group of the free group with the Graev topology by the commutator subgroup is the free abelian topological group.)

PROPOSITION. *Let X be any Tychonoff spaces with distinguished point e and $F(X)$ the free group on the set $X \setminus \{e\}$ which has e as its identity element. Let τ_G be the Graev topology on $F(X)$ with respect to X. If τ is any sequential group topology on $F(X)$ which is finer than τ_G, and induces a nondiscrete topology on X, then $(F(X), \tau)$ has S_ω as a closed subspace. Hence $(F(X), \tau)$ is not a metrizable space or even a Fréchet space. In particular, $(F(X), \tau_G)$ and the (Graev) free topological group on X are not Fréchet.*

THEOREM 1. *If A and B are nontrivial topological groups not both discrete such that $A \amalg B$ is sequential, then $A \amalg B$ is sequential of order ω_1. In particular, $A \amalg B$ is not metrizable or a Fréchet space.*

PROOF. Let $K(A, B)$ be the cartesian subgroup of $A \amalg B$: that is, the kernel of the homomorphism $A \amalg B \to A \times B$. Then $K(A, B)$ is algebraically a free group on the set $\{a^{-1}b^{-1}ab\colon a \in A \setminus \{e\}, b \in B \setminus \{e\}\}$. Graev [**6**] showed that the induced topology on $K(A, B)$ is finer than the Graev topology on the free group on a space X, where X is a continuous one-to-one image of the subspace $Y = \{a^{-1}b^{-1}ab\colon a \in A \setminus \{e\}, b \in B \setminus \{e\}\}$ of $A \amalg B$. (See the proof of Theorem 1 of [**8**].) As $K(A, B)$ is a closed subgroup of $A \amalg B$ it is sequential. Note that $K(A, B)$ is not discrete since A and B not both discrete and the maps $A \to K(A, B)$ and $B \to K(A, B)$ respectively given by $a \mapsto a^{-1}b_1^{-1}ab_1$ and $b \mapsto a_1^{-1}b^{-1}a_1b$ are continuous and one-to-one for fixed $a_1 \in A \setminus \{e\}$ and $b_1 \in B \setminus \{e\}$. Thus $K(A, B)$ satisfies the conditions of the above Proposition and so has a closed subspace homeomorphic to S_ω. Thus $A \amalg B$ has a closed subspace homeomorphic to $S_{\omega'}$ as required. □

COROLLARY. *Let A and B be nontrivial topological groups having a common closed normal subgroup N which is not an open subgroup of A. If $A \amalg_N B$ is sequential, then it is sequential of order ω_1.*

PROOF. Let γ be the canonical open continuous homomorphism of $A \amalg_N B$ onto $A/N \amalg B/N$. As $A \amalg_N B$ is sequential, so too is its quotient group $A/N \amalg B/N$. Noting that A/N is not discrete, the above theorem implies that $A/N \amalg B/N$ is sequential of order ω_1. Since γ is a quotient mapping this implies $A \amalg_N B$ is sequential of order ω_1. □

REMARK. The condition that N not be open in A cannot be deleted. For example, if $A = B = T \times \mathbf{Z}_2$ and $N = T$, where $\mathbf{Z}_2$ is the discrete cyclic group with two elements and T is the circle group with the usual compact metric group topology, then by [**9**, Proposition 4], $A \amalg_N B$ is homeomorphic to $T \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times D$, where D is a discrete group. This is clearly metrizable and hence of sequential order 1.

REMARK. We have improved upon Theorem 2 of Morris [**10**] in two different ways. First our amalgamated subgroup N is a normal subgroup rather than a central subgroup. Second, we consider sequential conditions rather than metrizability.

In Morris and Thompson [**14**] it is shown that if G is a subgroup of a free topological group and G is a sequential space, then it is sequential of order ω_1 or is discrete. This suggests the following

OPEN QUESTION. Let G be a nondiscrete closed subgroup of the free product of two nontrivial topological groups A and B such that G is not contained in a conjugate of A or a conjugate of B. If G is a sequential space is it sequential of order ω_1?

We note that the answer to this question is in the affirmative if A and B are both k_ω-groups and G is metrizable. This is so because $A \amalg B$ is then a k_ω-group as is its closed subgroup G. But as every metrizable k_ω-space is locally compact, G is locally compact. It then follows from [**13**] that G is either discrete or contained in a conjugate of A or a conjugate of B.

REFERENCES

1. A. V. Arhangel'skiĭ and S. P. Franklin, *Ordinal invariants for topological spaces*, Michigan Math. J. **15** (1968), 313–320.
2. Ryszard Engelking, *General topology*, Mathematical Monographs, 60, PWN, Warsaw, 1977.
3. S. P. Franklin, *Spaces in which sequences suffice*, Fund. Math. **57** (1965), 107–115.
4. ——, *Spaces in which sequences suffice*. II, Fund. Math. **61** (1967), 51–56.
5. M. I. Graev, *Free topological groups*, Izv. Akad. Nauk SSSR Ser. Mat. **12** (1948), 279–324 (Russian); English transl., Amer. Math. Soc. Transl., no. 35 (1951); reprint, Amer. Math. Soc. Transl. (1) **8** (1962), 305–364.
6. ——, *On free products of topological groups*, Izv. Akad. Nauk SSSR Ser. Mat. **14** (1950), 343–354. (Russian)
7. J. P. L. Hardy, S. A. Morris, and H. B. Thompson, *Applications of the Stone Čech compactification to free topological groups*, Proc. Amer. Math. Soc. **55** (1976), 160–164.
8. M. S. Khan and S. A. Morris, *Free products of topological groups with central amalgamation*. I, Trans. Amer. Math. Soc. **273** (1982), 405–416.
9. ——, *Free products of topological groups with central amalgamation*. II, Trans. Amer. Math. Soc. **273** (1982), 417–432.
10. S. A. Morris, *Metrizability of free products of topological groups*, Bull. Malaysian Math. Soc. **6** (1983), 7–11.
11. ——, *Free abelian topological groups*, Categorical Topology (Proc. Conf., Toledo, Ohio, 1983), Heldermann Verlag, Berlin, 1984, pp. 375–391.
12. S. A. Morris and P. Nickolas, *Locally invariant topologies on free groups*, Pacific J. Math. **103** (1982), 523–537.
13. ——, *Locally compact group topologies on an algebraic free product of groups*, J. Algebra **38** (1976), 393–397.
14. S. A. Morris and H. B. Thompson, *Metrizability of subgroups of free topological groups*, Bull. Austral. Math. Soc. **33** (1986), 103–112.
15. E. T. Ordman and B. V. Smith-Thomas, *Sequential conditions and free topological groups*, Proc. Amer. Math. Soc. **79** (1980), 319–326.

DEPARTMENT OF MATHEMATICS, LA TROBE UNIVERSITY, BUNDOORA, VICTORIA 3083, AUSTRALIA

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF QUEENSLAND, ST. LUCIA, QUEENSLAND 4067, AUSTRALIA (Current address of H. B. Thompson)

Current address (S. A. Morris): Department of Mathematics, Statistics and Computing Science, The University of New England, Armidale, N.S.W. 2351, Australia