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THE CIRCLE GROUP

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We prove the following theorem: if G is a locally compact Hausdorff group such that each of its proper closed subgroups has only a finite number of closed subgroups, then G is topologically isomorphic to the circle group.

Introduction

Armacost [1] gives various properties of the circle group, T, which characterize it in the class of non-discrete locally compact Hausdorff abelian groups. In particular, he records the following two:

(i) every proper closed subgroup is finite;

(ii) every proper closed subgroup is of the form $\{g : g^n = 1\}$, where n is any non-negative integer and 1 denotes the identity element.

We point out that both of these are special cases of the following property:

(iii) every proper closed subgroup has only a finite number of closed subgroups.

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We show that property (iii) characterizes T not only in the class of non-discrete locally compact Hausdorff abelian groups but also in the class of non-discrete locally compact Hausdorff groups.

Results

LEMMA 1. *If G is a compact Hausdorff abelian group with only a finite number of closed subgroups, then G is finite.*

Proof. By duality, the discrete dual group, $\hat{G}$, has only a finite number of quotient groups, and hence has only a finite number of subgroups. Clearly this implies that $\hat{G}$ is finitely generated. So $\hat{G}$ is algebraically ismorphic to a finite product of cyclic groups. Further, $\hat{G}$ does not have the discrete group of integers, Z, as a subgroup as Z has an infinite number of subgroups. Thus G is a finite group. Hence its dual group, G, is also finite. □

LEMMA 2. *If G is a totally disconnected locally compact topological group such that every proper closed subgroup has only a finite number of closed subgroups, then G is discrete.*

Proof. By [3, Theorem 7.7] G has a basis, $\{H_i : i \in I\}$, at 1 where I is an index set, and each H_i is a compact open proper subgroup. Let H be any proper compact open subgroup of G. Then $\{H \cap H_i : i \in I\}$ is an open basis at 1 for G. But H has only a finite number of closed subgroups. So $A = \cap_{i \in I}(H \cap H_i)$ is the smallest open subgroup of H. Indeed, A is the smallest open neighbourhood of 1. Since G and H are Hausdorff, this implies that $A = \{1\}$. Hence $\{1\}$ is open in H and G. Thus G is discrete. □

THEOREM. *Let G be a non-discrete locally compact Hausdorff topological group such that every proper closed subgroup has only a finite number of closed subgroups. Then G is topologically isomorphic to T.*

Proof. Let $C(G)$ be the connected component of the identity of G. By Lemma 2, $C(G) \neq \{1\}$. Clearly $C(G)$ does not have a closed subgroup topologically isomorphic to R, the topological group of all real numbers, as R has a proper closed subgroup Z which has an infinite number of closed subgroups. By the Iwasawa Structure Theorem [4, p.118], then, $C(G)$ is compact. So by the Peter-Weyl Theorem [4, pp.62-65], $C(G)$ has a closed normal subgroup N such that $C(G)/N$ is a compact connected Lie group (indeed a closed subgroup of a unitary group). Further, by [2, p.159], each $x \in C(G)/N$ lies in a closed subgroup A_x topologically isomorphic to a torus T^n, for some positive integer n. But T, and hence A_x, has an infinite number of closed subgroups. So if ϕ is the canonical map of $C(G)$ onto $C(G)/N$, then $\phi^{-1}(A_x)$ has an infinite number of closed subgroups. Even if $N = \{1\}$, this implies $\phi^{-1}(A_x) = G$. So $G = C(G)$. Further, G/N is topologically isomorphic to T^n. This is true for all closed subgroups, N_i, $i \in I$, such that G/N_i is a Lie group. By the Peter-Weyl Theorem, G is topologically isomorphic to a subgroup of $\Pi_{i\in I} G/N_i$, which is a product of tori. Hence G is abelian, But, then, N is a compact Hausdorff abelian group with only a finite number of closed subgroups. By Lemma 1, N is finite discrete. So G/N topologically isomorphic to T^n implies G/N is locally isomorphic to T^n. As G is compact and connected, this implies that G is topologically isomorphic to T^n [4, Theorem 8]. But every proper closed subgroup of G has only a finite number of closed subgroups so, $n = 1$. □

We conclude with two corollaries which follow immediately from our theorem. The first was recently proved. The second appeared in [1] under the additional assumption that G was abelian.

COROLLARY 1. [5] *Let G be a non-discrete locally compact Hausdorff group such that each of its proper closed subgroups is finite. Then G is topologically isomorphic to T.* □

COROLLARY 2. *Let G be a non-discrete locally compact Hausdorff group such that its proper closed subgroups are $\{g : g^n = 1\}$, for n ranging over the set of all positive integers. Then G is topologically isomorphic to T.* □

References

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