TRINITY ... A TALE OF THREE CARDINALS

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Dedicated to Igor Kluvanek

1. INTRODUCTION

In this paper we discuss three cardinal numbers associated with a topological group G : the weight of G, $\omega(G)$, the local weight, $\omega_0(G)$, and $\theta(G)$, the least cardinal of a family of open sets whose intersection is a singleton. It is clear that $\theta(G) \leqslant \omega_0(G) \leqslant \omega(G)$. We give necessary and sufficient conditions for $\theta(G) = \omega_0(G) = \omega(G)$. In particular they are equal for all σ-compact locally compact Hausdorff groups.

The following notation will be used throughout the paper. If G is a topological group, we denote

(a) the minimal cardinality of a family of open sets having as intersection the identity, 1, in G by $\theta(G)$;

(b) the minimal cardinality of an open basis for G at 1 by $\omega_0(G)$;

(c) the minimal cardinality of an open basis for G by $\omega(G)$.

If H is a topological subgroup of G, we write $H \leqslant G$.

Note that if $H \leqslant G$, then $\theta(H) \leqslant \theta(G)$, $\omega_0(H) \leqslant \omega_0(G)$, and $\omega(H) \leqslant \omega(G)$.

PROPOSITION 1 *If* G *is any topological group then*
$\theta(G) \leqslant \omega_0(G) \leqslant \omega(G)$.

Proof Clearly $\theta(G) \leqslant \omega_0(G)$ and $\omega_0(G) \leqslant \omega(G)$. So
$\theta(G) \leqslant \omega_0(G) \leqslant \omega(G)$. //

We note here that if an infinite Hausdorff non-discrete topological group, G, satisfies the second axiom of countability, then $\theta(G) = \omega_0(G) = \omega(G) = \aleph_0$. Thus if G is an infinite compact metrizable group, then $\theta(G) = \omega(G) = \aleph_0$.

DEFINITION Let $U(n)$, $n \in \mathbb{N}$, be the compact group of $n \times n$ unitary matrices, and define $\mathfrak{U} = \prod_{n=1}^{\infty} U(n)$.

As $\mathfrak{U}$ is compact and metrizable $\omega(\mathfrak{U}) = \theta(\mathfrak{U}) = \omega_0(\mathfrak{U}) = \aleph_0$.

2. COMPACT GROUPS

We use the following refinement of the Embedding Lemma, ([6],P.116) in the proof of Lemma 3. It's proof is analogous to the usual proof.

LEMMA 2 *Let* $\{(Y_i,\tau_i) \mid i \in I\}$ *be a family of Hausdorff spaces, and for each* $i \in I$, *let* f_i *be a mapping of a Hausdorff space* (X,τ) *into* (Y_i,τ_i). *Let* $e : (X,\tau) \to \prod_{i\in I} (Y_i,\tau_i)$ *be defined by* $e(x) = \prod_{i\in I} f_i(x)$, *for each* $x \in X$. *Then* e *is a homeomorphism of* (X,τ) *onto the space* $(e(X),\tau')$ *where* τ' *is the subspace topology, if*

(i) *each* f_i *is continuous, and*

(ii) *given* $x \in X$ *and any closed set* A *not containing* x, *there is a finite subset* $\{i_1, i_2, \ldots, i_n\}$ *of* I *such that the map* $F = f_{i_1} \times f_{i_2} \times \ldots \times f_{i_n} : X \longrightarrow \prod_{j=1}^{n} (Y_{i_j}, \tau_{i_j})$ *satisfies* $F(x) \notin \overline{F(A)}$.

LEMMA 3 *Let* G *be a topological group and* $\{H_i \mid i \in I\}$ *an infinite family of Hausdorff groups such that* G *is topologically isomorphic to a subgroup of the product* $\prod_{i \in I} H_i$. *Then there is a subset* J *of* I, *with* $\text{card } J = \omega_0(G)$, *such that* G *is topologically isomorphic to a subgroup of* $\prod_{i \in J} H_i$.

Proof Without loss of generality, consider G to be a subgroup of $\prod_{i \in I} H_i$. Let $B = \{B_k \mid k \in K\}$ be a basis for G at the identity, 1, such that $\text{card } K = \omega_0(G)$. For each $k \in K$ there exists an O_k such that $O_k \cap G \subseteq B_k$ where $O_k = O_{k_1} \times O_{k_2} \times \ldots \times O_{k_n} \times \prod_{i \in I \setminus \{k_1, k_2, \ldots k_n\}} H_i$ is a member of the natural basis for $\prod_{i \in I} H_i$ at the identity. For each $k \in K$ put $J_k = \{k_1, k_2, \ldots, k_n\}$ and $J = \bigcup_{k \in K} J_k$. Then, as each J_k is finite, $\text{card } J = \text{card } K = \omega_0(G)$.

Let $P : \prod_{i \in I} H_i \rightarrow \prod_{i \in J} H_i$ be the natural projection mapping. We need to show $P : G \rightarrow P(G)$ is a homeomorphism. As each $p_i : G \rightarrow H_i$ given by $p_i(x) = p_i(\prod_{i \in I} x_i) = x_i$, is continuous, condition (i) of the Embedding Lemma is satisfied. To see condition (ii) holds, we need consider only the identity 1 and any closed set A in G such that $1 \notin A$. Then $1 \in G \setminus A$ which is open, and so there is a $B_k \in B$ such that $1 \in B_k \cap G$. Therefore there is a basic open neighbourhood O_k such that $1 \in O_k \cap G$; that is

$1 \in (O_{k_1} \times O_{k_2} \times \ldots \times O_{k_n} \times \prod_{i \in I \setminus \{k_1, k_2, \ldots, k_n\}} H_i) \cap G$. Define

$F : G \to \prod_{j=1}^{n} H_{k_j}$ by $F(x) = \prod_{j=1}^{n} P_{k_j}(x)$, for $x \in G$. Then

$F(1) \in O_{k_1} \times O_{k_2} \times \ldots \times O_{k_n}$ which is open and $F(A) \cap (O_{k_1} \times O_{k_2} \times \ldots \times O_{k_n}) = \emptyset$ which implies $\overline{F(A)} \cap (O_{k_1} \times O_{k_2} \times \ldots \times O_{k_n}) = \emptyset$. Hence $F(1) \notin \overline{F(A)}$, and so by our Embedding Lemma, P is a homeomorphism of G onto $P(G)$. As P is also a homomorphism we have that G is topologically isomorphic to $P(G)$, a subgroup of $\prod_{i \in J} H_i$. //

The countable case of the above result was used by Brooks, Morris and Saxon [2, Corollary 6].

Using a similar argument to the proof of Lemma 3, we obtain a stronger result for compact groups.

LEMMA 4 *Let* G *be a compact group and* $\{H_i \mid i \in I\}$ *an infinite family of Hausdorff groups such that* G *is topologically isomorphic to a subgroup of the product* $\prod_{i \in I} H_i$. *Then there is a subset* J *of* I, *with* $\text{card } J = \theta(G)$, *such that* G *is topologically isomorphic to a subgroup of* $\prod_{i \in J} H_i$.

Proof Again, consider G to be a subgroup of $\prod_{i \in I} H_i$, and let $\Phi(G) = \{U_k \mid k \in K\}$ be a family of open sets of G such that $\text{card } \Phi(G) = \theta(G)$ and $\bigcap_{k \in K} U_k = \{1\}$. For each $k \in K$ there is an open set O_k such that $O_k \cap G \subseteq U_k$ where

$O_k = O_{k_1} \times O_{k_2} \times \ldots \times O_{k_n} \times \prod_{i \in I \setminus \{k_1, k_2, \ldots k_n\}} H_i$ is a member of the natural basis for $\prod_{i \in I} H_i$ at the identity. For each $k \in K$ put $J_k = \{k_1, k_2, \ldots, k_n\}$ and $J = \bigcup_{k \in K} J_k$. Then card J = card $K = \theta(G)$.

Let $P : \prod_{i \in I} H_i \to \prod_{i \in J} H_i$ be the natural projection mapping. Then $P : G \to \prod_{i \in J} H_i$ is a continuous injective homomorphism. As G is compact, G is topologically to $P(G)$, from which the result follows. //

The next lemma is an immediate consequence of the Peter-Weyl Theorem ([7], P.62).

LEMMA 5 *If* G *is a compact Hausdorff group, then it is topologically isomorphic to a subgroup of a product of copies of the group* $\mathfrak{U}$.

THEOREM 1 [3, 28.58] *Let* G *be an infinite compact Hausdorff group. Then* $\theta(G) = \omega_0(G) = \omega(G)$.

Proof By Lemma 5, we can, without loss of generality, assume that G is a subgroup of $\mathfrak{U}^{\text{card } I}$, for some index set I. But using Lemma 4 we have that G is topologically isomorphic to a subgroup of $\mathfrak{U}^{\theta(G)}$.

$$\begin{aligned} \text{So } \omega(G) &\leqslant \omega(\mathfrak{U}^{\theta(G)}) \\ &= \max\{\omega(\mathfrak{U}), \theta(G)\} \\ &= \max\{\aleph_0, \theta(G)\} \\ &= \theta(G), \text{ as } \theta(G) \text{ is infinite.} \end{aligned}$$

But $\theta(G) \leqslant \omega(G)$ from Proposition 1. Thus $\theta(G) = \omega(G)$, from which it follows that $\omega_0(G) = \theta(G) = \omega(G)$. //

Hulanicki [3] proved that $\text{card } G = 2^{\theta(G)}$ for G, any infinite compact Hausdorff group, or any infinite connected locally compact Hausdorff group. Elsewhere we shall give quite a different proof of a more general result. Here we point out a corollary to this result and Theorem 1.

THEOREM 2 [3, 28.58] *Let* G *be any infinite compact Hausdorff group. Then* $\text{card } G = 2^{\theta(G)} = 2^{\omega_0(G)} = 2^{\omega(G)}$.

3. ALMOST CONNECTED GROUPS

DEFINITION A locally compact Hausdorff group is said to be *almost connected* if the group G/G_0 is compact, where G_0 is the connected component of the identity. (See [1].)

Of course, the class of almost connected groups includes the class of compact Hausdorff groups and the class of connected locally compact Hausdorff groups.

THEOREM 3 *Let* G *be any infinite almost connected group. Then* $\theta(G) = \omega_0(G) = \omega(G)$ *and* $\text{card } G = 2^{\theta(G)} = 2^{\omega_0(G)} = 2^{\omega(G)}$.

Proof By Mostert ([7], Theorem 8) G is homeomorphic to $G_0 \times G/G_0$. The Iwasawa Structure Theorem ([6], p.118) says that the connected locally compact Hausdorff group G_0 is homeomorphic to $\mathbb{R}^n \times K$, where K is a compact group, $\mathbb{R}$ is the topological group of real numbers with the usual topology, and n is a non-negative integer. As G/G_0 is compact, we have that G is homeomorphic to $\mathbb{R}^n \times K'$ where K' is the compact Hausdorff group $K \times G/G_0$.

If K' is finite, then clearly $\theta(G) = \omega_0(G) = \omega(G) = \aleph_0$, and $\text{card } G = 2^{\aleph_0}$.

If K' is infinite, then $\theta(G) = \theta(\mathbb{R}^n \times K') = \theta(\mathbb{R}^n) \times \theta(K')$. Since $\theta(\mathbb{R}^n) = \aleph_0$ we have that $\theta(G) = \theta(K')$. Similarly, $\omega_0(G) = \omega_0(K')$ and $\omega(G) = \omega(K')$. Then, by Theorem 1, we have $\theta(G) = \omega_0(G) = \omega(G)$.

$$\begin{aligned}\text{Further,}\quad \text{card } G &= \text{card } \mathbb{R}^n \times \text{card } K' \\ &= 2^{\aleph_0} \times 2^{\theta(K')} \\ &= 2^{\aleph_0 + \theta(K')} \\ &= 2^{\theta(K')}.\end{aligned}$$

Hence, $\text{card } G = 2^{\theta(G)} = 2^{\omega_0(G)} = 2^{\omega(G)}$. //

4. THE GENERAL CASE

For G, any topological group, we denote the least cardinality of a family of compact sets whose union is G by $\gamma(G)$.

LEMMA 6 *Every locally compact Hausdorff group has an open almost connected subgroup.*

Proof Let G be any locally compact Hausdorff group and let G_0 be the component of the identity. Let $f : G \to G/G_0$ be the quotient mapping. Then the quotient group G/G_0 is a locally compact totally disconnected group and so has a basis of compact open subgroups, ([7],p.21). Take one such compact open subgroup, K. Then $f^{-1}(K) = H$ is an open subgroup of G. As H is open and therefore closed, $G_0 \subseteq H$, and so $H_0 = G_0$. This implies $H/H_0 = H/G_0 = K$. Hence H is a locally compact Hausdorff group, and H/H_0 is compact, from which the result follows. //

THEOREM 4 *Let* G *be any infinite locally compact Hausdorff group. Then* (i) $\omega_0(G) = \theta(G)$; (ii) $\omega(G) = \max\{\omega_0(G), \gamma(G)\}$ and (iii) $\text{card } G = \max\{2^{\omega_0(G)}, \gamma(G)\}$.

Proof (i) Let H be an open almost connected subgroup of G. Then $\omega_0(H) = \theta(H)$ by Theorem 3. We show that $\omega_0(G) = \omega_0(H)$ and $\theta(G) = \theta(H)$, from which the result will follow.

Let B_0 be a basis for H at the identity with $\text{card } B_0 = \omega_0(H)$. Then B_0 is also a basis for G at the identity. So $\omega_0(G) \leqslant \omega_0(H)$, and hence $\omega_0(G) = \omega_0(H)$.

Let $\Phi(H)$ be a family of open sets in H whose intersection is the identity. Then $\Phi(H)$ is also a family of open sets in G whose intersection is the identity, as H is open. So $\theta(G) \leqslant \theta(H)$, and hence $\theta(G) = \theta(H)$.

(ii) If G is compact $\omega(G) = \omega_0(G)$ from Theorem 3, and $\gamma(G) = 1$, which implies $\omega(G) = \max\{\omega_0(G), \gamma(G)\}$. So assume G is non-compact. Let $\{g_i \mid i \in I\}$ be a complete set of coset representatives of H in G, and let card $I = m$. We show firstly that $\omega(G) = \max\{\omega(H), m\}$. Let $\mathcal{B}$ be a basis for H. It is clear that $\{g_i B \mid B \in \mathcal{B}, i \in I\}$ is a basis for G as H is open. Thus $\omega(G) \leqslant \max\{\omega(H), m\}$. We know that $\omega(H) \leqslant \omega(G)$, and, as each coset is open and must contain a basic open set of G, $\omega(G) \geqslant m$. Hence $\omega(G) = \max\{\omega(H), m\}$.

As H is almost connected, it is homeomorphic to $\mathbb{R}^n \times K$, where K is a compact group and $n \in \mathbb{N}$. Therefore $\gamma(H) \leqslant \aleph_0$. Let $\{A_n \mid n \in \mathbb{N}\}$ be a family of compact sets whose union is H. Then $\{g_i A_n \mid i \in I, n \in \mathbb{N}\}$ is a family of compact sets whose union is G, and therefore $\gamma(G) \leqslant \max\{\aleph_0, m\}$. Let $\{K_j \mid j \in J\}$ be a family of compact sets whose union is G and with card $J = \gamma(G)$. Then each K_j, being compact, is contained in the union of a finite number of cosets; that is, $K_j \subseteq \bigcup_{k=1}^{m_j} g_{i_k} H$ for $m_j \in \mathbb{N}$. So $\gamma(G) = \text{card } J \geqslant m$. Now, clearly, $\gamma(G) \geqslant \aleph_0$, and so we get $\gamma(G) = \max\{\aleph_0, m\}$.

Finally, we have
$$\begin{aligned} \omega(G) &= \max\{\omega(H), m\} \\ &= \max\{\omega_0(G), m\}, \text{ as } \omega(H) = \omega_0(H) = \omega_0(G) \\ &= \max\{\omega_0(G), m, \aleph_0\}, \text{ as } \omega(G) \text{ is infinite} \\ &= \max\{\omega_0(G), \gamma(G)\} . \end{aligned}$$

(iii) If G is compact we already have that

card G = $2^{\omega_0(G)}$ ={max $2^{\omega_0(G)}$, $\gamma(G)$} from Theorem 2, so again assume G is non-compact. Then

card G = card H.m = max{$2^{\omega_0(H)}$, m} = max{$2^{\omega_0(G)}$, $\gamma(G)$}. //

We note that Hulanicki's Fundamental lemma is a corollary to the above theorem.

COROLLARY 1 ([4], p.67) If G is an infinite locally compact Hausdorff group, then card G $\geqslant 2^{\theta(G)}$. //

COROLLARY 2 Let G be an infinite locally compact Hausdorff group. Then the following are equivalent

(i) $\omega(G) = \omega_0(G)$; (ii) $\gamma(G) \leqslant \omega_0(G)$. //

COROLLARY 3 ([3], p.100) If G is an infinite σ-compact locally compact Hausdorff group, then $\omega(G) = \omega_0(G) = \theta(G)$. //

COROLLARY 4 ([4], p.69) If the locally compact Hausdorff group, G, is $2^{\theta(G)}$-compact, then card G = $2^{\theta(G)}$.

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