A CHARACTERIZATION OF THE TOPOLOGICAL GROUP OF p-ADIC INTEGERS

SIDNEY A. MORRIS AND SHEILA OATES-WILLIAMS

Abstract

It is proved that a compact Hausdorff group is topologically isomorphic to the topological group of p-adic integers, for some prime number p, if and only if all of its non-trivial proper closed subgroups are topologically isomorphic.

Introduction and preliminaries

Armacost [**1**] gives characterizations of some important locally compact abelian groups in terms of their closed subgroups. One of these (families of) groups is Δ_p, the topological group of p-adic integers, where p is any prime number. (See [**4**, §10; **2**] for a description of Δ_p.) If $G = \Delta_p$, then its non-trivial proper closed subgroups are $p^n G$, where n ranges over the set of positive integers. Further, Δ_p is a compact Hausdorff totally disconnected group and each of its closed subgroups is open, of finite index and topologically isomorphic to Δ_p. Armacost [**1**] proved that a compact Hausdorff abelian topological group G is topologically isomorphic to Δ_p, for some prime number p, if and only if all of its non-trivial proper closed subgroups are topologically isomorphic. We prove that the assumption that G is abelian can be omitted.

In what follows the identity of a group is denoted by 1, $[g, h]$ denotes $ghg^{-1}h^{-1}$ and C_n the cyclic group of order n, where n is a positive integer. If G is any group then $Z(G)$ denotes the centre of G. For any subset S of G, $\mathrm{gp}\{S\}$ denotes the subgroup of G generated by S and $\overline{\mathrm{gp}\{S\}}$ the closure of $\mathrm{gp}\{S\}$. The circle group is denoted by $\mathbf{T}$.

Results

LEMMA. *Let G be a torsion-free group with its centre $Z(G)$ having finite index in G. If there exists a prime number p such that every proper subgroup of G which contains $Z(G)$ is algebraically isomorphic to Δ_p, then G is abelian.*

Proof. Suppose that G is non-abelian. Then the factor group $K = G/Z(G)$ is not a cyclic group. Since every factor group of Δ_p is a finite cyclic p-group, K is a non-cyclic p-group all of whose proper subgroups are cyclic. Clearly the only abelian group with this property is $C_p \times C_p$ and it follows from [**6**, p. 149, Theorem 17] that the only non-abelian group with this property is the quaternion group of order eight. So there are two cases to consider.

(I) Assume K is algebraically isomorphic to $C_p \times C_p$. Then

$$G = \mathrm{gp}\{a, b, Z(G) : a^p = c,\quad b^p = d,\quad [a, b] = e,\quad c, d, e \in Z(G)\}.$$

Received 21 March 1986.

1980 *Mathematics Subject Classification* 22C05.

Now if $c \in (Z(G))^p$, then $c = f^p$, where $f \in Z(G)$ and so $(af^{-1})^p = 1$, while $af^{-1} \neq 1$. Thus af^{-1} is a torsion element of G, which is impossible. Hence $c \notin (Z(G))^p$. Similarly $d \notin (Z(G))^p$. Thus, $c, d \in Z(G) \backslash (Z(G))^p$. As $Z(G)$ is algebraically isomorphic to Δ_p, $Z(G)/(Z(G))^p$ is algebraically isomorphic to C_p. So we must have

$$d = c^s g^p, \quad 1 \leqslant s \leqslant p-1, \quad g \in Z(G).$$

Then, by [6, p. 81 (10)],

$$(a^{p-s}b)^p = (a^p)^{p-s} b^p [b, a^{p-s}]^{p(p-1)} = c^{p-s} d e^{-(p-s)p(p-1)/2} = k^p$$

where $k \in Z(G)$, if $p > 2$. Once again we have a torsion element of G, which is a contradiction.

If $p = 2$, the result follows if $e \in (Z(G))^2$. Otherwise $d = cg^2$, $e = ch^2$, for $g, h \in Z(G)$, and $(ab)^2 = c \cdot cg^2 \cdot c^{-1}h^{-2} = a^2k^2$, where $k \in Z(G)$. So $a^{-1}bab = k^2$; that is, $a^{-1}ba = k^2b^{-1}$. Thus $a^{-1}b^2a = k^4b^{-2}$. But b^{-2} is central, so $b^2 = k^4b^{-2}$; that is, $b^4 = k^4$. So $(bk^{-1})^4 = 1$, and again we have a torsion element. Hence K is not algebraically isomorphic to $C_p \times C_p$.

(II) Assume that K is algebraically isomorphic to Q, the quaternion group of order eight. Then $G = \mathrm{gp}\{a, b, Z(G)\}$, where, since Q has generating relations $a^2 = b^2 = (ab)^2$, we have $a^2 = b^2c$, and $a^2 = (ab)^2d$, for $c, d \in Z(G)$. But $a^2 = b^2c$ implies that both a and b commute with a^2. So a^2 is central in G; that is, $a^2 \in Z(G)$. Hence K cannot be algebraically isomorphic to Q.

Thus G must be abelian.

THEOREM. *Let G be an infinite compact Hausdorff group. Then the following are equivalent:*

(i) *G is topologically isomorphic to Δ_p;*
(ii) *all non-trivial closed subgroups of G are topologically isomorphic to G;*
(iii) *all non-trivial proper closed subgroups of G are topologically isomorphic.*

Proof. By [2, Theorem 1.10], (i) implies (ii), while (ii) clearly implies (iii). So it suffices to prove that (iii) implies (i).

Assume (iii) is true. Let g be any element of G, and let S_g denote the closure of $\mathrm{gp}\{g\}$. Then S_g is a compact Hausdorff abelian group having the property that all of its non-trivial proper closed subgroups are topologically isomorphic. So by [2, Theorem 1.10], S_g is topologically isomorphic to Δ_p. Therefore, from our assumption, all non-trivial proper closed subgroups of G are topologically isomorphic to Δ_p. This implies, in particular, that G is torsion-free.

Let $C(G)$ be the component of 1. As Δ_p is totally disconnected, while $C(G)$ is connected, either $C(G) = G$ or $C(G) = \{1\}$.

If $C(G) = G$, then G is a compact connected Hausdorff group. So, by the Peter–Weyl Theorem [5, pp. 62–65], G has a closed normal subgroup N such that G/N is a connected Lie group (indeed a closed subgroup of a unitary group). Further, by [3, p. 159] each $x \in G/N$ lies in a closed subgroup A_x topologically isomorphic to a torus $\mathbf{T}^n$, for some positive integer n. Let ϕ be the quotient mapping of G onto G/N. Then $\phi^{-1}(A_x)$ is a closed non-trivial subgroup of G. If $\phi^{-1}(A_x) \neq G$, then it is topologically isomorphic to Δ_p. But then $\mathbf{T}^n$ would be a quotient group of Δ_p, which is impossible as all of the closed subgroups of Δ_p have finite index. So $\phi^{-1}(A_x) = G$; that is, G/N is topologically isomorphic to $\mathbf{T}^n$. If $\{N_i : i \in \mathrm{I}\}$ is the family of all closed normal subgroups of G such that G/N is a Lie group, then the Peter–Weyl Theorem implies that G is topologically isomorphic to a subgroup of $\Pi_{i\in I} G/N_i$. But as each G/N_i is topologically isomorphic to $\mathbf{T}^{n_i}$ it is abelian, and hence G too is abelian. Then by (iii) and [**2**, Theorem 1.10], G is topologically isomorphic to Δ_p. This is a contradiction, since G was assumed to be connected.

Therefore $C(G) = \{1\}$; that is, G is totally disconnected. Then, by [**4**, Theorem 7.7] G has a basis at the identity consisting of compact open normal proper non-trivial subgroups. Each of these subgroups is topologically isomorphic to Δ_p. Let B be any one of these subgroups and E any non-trivial proper closed subgroup of G. Then $B \cap E$ is a closed subgroup of B. But B is topologically isomorphic to Δ_p, and so each of its non-trivial closed subgroups is also open in B. If $B \cap E = \{1\}$, then $\{1\}$ would be open in E, and so E would be discrete. But this is not so, because E is topologically isomorphic to Δ_p. Hence $B \cap E$ is open in B, and hence also in G. As E is a union of cosets of $B \cap E$, E is open in G. Because G is compact, this implies E has finite index in G.

To sum up so far, we have shown that G is torsion-free and every non-trivial proper closed subgroup is open, has finite index in G, and is topologically isomorphic to Δ_p.

Suppose that g and h belong to G and are such that $gh \neq hg$. Put $S_g = \overline{\mathrm{gp}\{g\}}$, $S_h = \overline{\mathrm{gp}\{h\}}$, $X = \overline{\mathrm{gp}\{g, h\}}$ and $f = S_g \cap S_h$. Then F, being the intersection of two open subsets of G, is open in G. As G is not discrete, F is non-trivial. As every element of F commutes with g and h, F is a subgroup of $Z(X)$. Hence $Z(X)$ is a non-trivial proper closed and open subgroup of X of finite index. Observe that any proper subgroup of X which contains $Z(X)$ is a union of cosets of $Z(X)$ and so is open (and closed) in X. Hence the subgroup is topologically isomorphic to Δ_p. Thus X satisfies the conditions of the Lemma, and so is abelian. But this is a contradiction. Therefore, for all g and h in G, $gh = hg$; that is, G is abelian. Then by (iii) and [**2**, Theorem 1.10], G is topologically isomorphic to Δ_p.

References

1. D. L. Armacost, 'Well-known LCA groups characterized by their closed subgroups', *Proc. Amer. Math. Soc.* 25 (1970) 625–629.
2. D. L. Armacost, *The structure of locally compact abelian groups* (Marcel Dekker, New York, 1981).
3. T. Bröcker and T. tom Dieck, *Representations of Lie groups* (Springer, New York, 1985).
4. E. Hewitt and K. A. Ross, *Abstract harmonic analysis I* (Springer, Berlin, 1963).
5. S. A. Morris, *Pontryagin duality and the structure of locally compact abelian groups* (Cambridge University Press, 1977).
6. H. J. Zassenhaus, *The theory of groups* (Chelsea Publishing Company, New York, 1974).

Department of Mathematics
La Trobe University
Bundoora, Victoria
Australia 3083

Department of Mathematics
University of Queensland
St. Lucia, Queensland
Australia 4067