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Locally compact topologies on abelian groups

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Abstract. It is shown that an abelian group admits a non-discrete locally compact group topology if and only if it has a subgroup algebraically isomorphic to the group of p-adic integers or to an infinite product of non-trivial finite cyclic groups. It is also proved that an abelian group admits a non-totally-disconnected locally compact group topology if and only if it has a subgroup algebraically isomorphic to the group of real numbers. Further, if an abelian group admits one non-totally-disconnected locally compact group topology then it admits a continuum of such topologies, no two of which yield topologically isomorphic topological groups.

Introduction. Soundararajan [**5**] and Corwin [**2**] proved that the group Δ_p of p-adic integers admits, up to isomorphism, one locally compact Hausdorff group topology. (See [**3**] for a description of the usual topology on Δ_p). Corwin remarks that 'Groups with this property seem to be fairly rare'. We show that any infinite abelian group which admits a locally compact Hausdorff group topology which is not totally disconnected also admits an uncountable number of locally compact Hausdorff group topologies no two of which give topologically isomorphic topological groups. Further, we show that an infinite abelian group admits a non-totally disconnected locally compact Hausdorff group topology if and only if it has a subgroup algebraically isomorphic to $\mathbf{R}$, the group of all real numbers. We then proceed to describe the class of abelian groups which admit a non-discrete locally compact Hausdorff group topology. This class consists of those groups which have a subgroup algebraically isomorphic either to Δ_p or to $\Pi_{i\in I} A_i$, where each A_i is algebraically isomorphic to a non-trivial finite cyclic group.

LEMMA 1. *Let I and J be subsets of the set of all prime numbers. Let E_I be the minimal divisible extension of the product group $\Pi_{p\in I}\Delta_p$ with the latter group as an open subgroup. Also let E_J be analogously defined. If D is any discrete abelian group and $\mathbf{R}\times E_I\times D$ is topologically isomorphic to $\mathbf{R}\times E_J\times D$, then $I = J$.*

Proof. Clearly it suffices to show that $p\in I$ implies $p\in J$. Now $p\in I$ implies that $\Delta_p \leqslant \mathbf{R}\times E_J\times D$. [We use the shorthand $A\leqslant B$ to mean that B has a subgroup topologically isomorphic to A.] Let ϕ_R, ϕ_E, and ϕ_D be the projection mappings of $R\times E_J\times D$ on to its factors $\mathbf{R}$, E_J, and D, respectively. As $\phi_R(\Delta_p)$ and $\phi_D(\Delta_p)$ are compact, they must both be finite. (Indeed the first one must be trivial.) Thus $\phi_E(\Delta_p)$ is an infinite compact Hausdorff group, and so is topologically isomorphic to Δ_p. (See corollary 1·3 of [**1**].) Thus $\Delta_p\leqslant E_J$. Now E_J has $\Pi_{q\in J}\Delta_p$ as an open subgroup. The intersection of this open subgroup with Δ_p yields an open subgroup of Δ_p; that is, a subgroup topologically isomorphic to Δ_p. Thus $\Delta_p\leqslant \Pi_{q\in J}\Delta_q$. As each Δ_q is torsion-free, the projection of Δ_p on to each factor Δ_q is either trivial or topologically isomorphic to Δ_p. Thus $\Delta_p \leqslant \Delta_q$, for some $q \in J$. But this implies $q = p$, and so $p \in J$, as required. · ∎

We shall use the term *LCA group* as an abbreviation for 'locally compact Hausdorff abelian topological group'.

THEOREM 1. *Let G be an infinite abelian group. Then the following are equivalent:*

(i) *G has a subgroup algebraically isomorphic to* $\mathbf{R}$*;*

(ii) *G is algebraically isomorphic to $A \times \mathbf{R}$, where A is an abelian group;*

(iii) *there exists a non-totally disconnected LCA group, H, which is algebraically isomorphic to G;*

(iv) *there exists a family $\{H_i : i \in L\}$ of non-totally disconnected LCA groups such that each H_i is algebraically isomorphic to G, H_i is not topologically isomorphic to H_j, for $i, j \in L$ and $i \neq j$, and L has cardinality, c, equal to that of the continuum.*

Proof. Firstly observe that (iv) trivially implies (iii) and (ii) trivially implies (i). Secondly, as $\mathbf{R}$ is divisible, (i) implies (ii). So it suffices to prove that (ii) implies (iv) and (iii) implies (i).

Assume that (ii) is true. Let I be any subset of the set of prime numbers and let E_I be as in Lemma 1. Then E_I is a torsion-free divisible abelian group of cardinality c, and so is algebraically isomorphic to $\mathbf{R}$. (See [**3**], theorems A·15, A·16, and A·14.) Put $H_I = \mathbf{R} \times E_I \times D$, where D is the group A with the discrete topology. As $\mathbf{R} \times \mathbf{R}$ is algebraically isomorphic to $\mathbf{R}$, we see that H_I is algebraically isomorphic to G. Lemma 1 implies that for distinct subsets I and J of the set of prime numbers H_I is not topologically isomorphic to H_J. Finally observe that there are c ways to choose a subset I of the set of prime numbers. So (iv) is true.

Now assume that (iii) holds. Let C be the component of 0 in H. As H is not totally disconnected, C is a non-trivial connected LCA group. So C is topologically isomorphic to $\mathbf{R}^n \times K$, where K is a compact connected group and n is a non-negative integer. If $n \neq 0$, then H contains $\mathbf{R}$ and so G contains a subgroup algebraically isomorphic to $\mathbf{R}$; that is, (i) is true. If $n = 0$, then K is a non-trivial compact connected abelian group. By theorem 25·23 of [**3**], K has a subgroup algebraically isomorphic to a restricted direct product of a continuum of copies of the group, $\mathbf{Q}$, of all rational numbers; that is, a subgroup algebraically isomorphic to $\mathbf{R}$. So (iii) implies (i). ∎

Remark. Theorem 1 is best possible in the sense that, if the continuum hypothesis is assumed, then §25·33 of [**3**] implies that if $\{H_i : i \in L\}$ is a family of LCA groups such that each H_i is algebraically isomorphic to $\mathbf{R}$ and H_i topologically isomorphic to H_j implies $i = j$; then L has cardinality not greater than c.

We now describe the abelian groups which admit a non-discrete locally compact Hausdorff group topology. This result contrasts with theorem 25·23 of [**3**], which is a characterization of those abelian groups which admit a compact Hausdorff group topology.

THEOREM 2. *Let G be an abelian group. Then the following are equivalent:*

(i) *There exists a prime number p, such that G has a subgroup which is algebraically isomorphic to Δ_p or $\prod_{i=1}^{\infty} A_i$, where each A_i is a non-trivial finite cyclic group.*

(ii) *There exists a non-discrete LCA group, H, which is algebraically isomorphic to G.*

Proof. Firstly assume that G has a subgroup algebraically isomorphic to Δ_p, for some p. Let $\mathscr{B}$ be an open basis at 0 for the usual topology on Δ_p. Then $\mathscr{B}$ is an open basis at 0 for a non-discrete locally compact Hausdorff group topology on G.

Next, assume that G has a subgroup algebraically isomorphic to $\prod_{i=1}^{\infty} A_i$. Put the discrete topology on each A_i and the (compact) product topology on $\prod_{i=1}^{\infty} A_i$. Then, as above, an open basis at 0 for this topology is also an open basis at 0 for a non-discrete locally compact Hausdorff group topology on G. So we have that (i) implies (ii).

Now assume that (ii) is true. If H is not totally disconnected then Theorem 1 implies that G has a subgroup algebraically isomorphic to $\mathbf{R}$. As $\mathbf{R}$ has a subgroup isomorphic to Δ_p for each p, we would have (i) is true. (Indeed $\mathbf{R}$ is, algebraically, the minimal divisible extension of the abstract group Δ_p.) So let H be totally disconnected. Then it has a basis at 0 of compact open subgroups. Let K be a compact open subgroup of H. If K is not a torsion group, then it has an element, x, of infinite order. So the closure of the subgroup generated by x is an infinite compact 0-dimensional monothetic group. By theorem 25·16 of [**3**], this monothetic group has a subgroup topologically isomorphic to Δ_p, for some prime number p, or to an infinite product of non-trivial finite cyclic groups. Finally, let K be a torsion group. As H is not discrete and K is an open subgroup, K must be infinite. But theorem 25·9 of [**3**] then says that the infinite compact torsion group K is topologically isomorphic to an infinite product of non-trivial finite cyclic groups. So (ii) does imply (i). ∎

Remarks. Theorem 2 is best possible in the sense that there exist abelian groups which admit only one non-discrete locally compact Hausdorff group topology. Each Δ_p is such a group.

A trivial consequence of Theorem 2 is that any LCA group of cardinality $< c$ is discrete. A little less trivial is the observation that a free abelian group does not admit a non-discrete locally compact Hausdorff group topology. More generally, a torsion-free LCA group which does not contain any Δ_p is discrete. This observation applies also to any product of infinite cyclic groups.

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