Math. Proc. Camb. Phil. Soc. (1986), **100**, 347
Printed in Great Britain

Free subgroups of free abelian topological groups

BY E. KATZ
Cleveland State University, Cleveland, OH 4415, *U.S.A.*

S. A. MORRIS
La Trobe University, Bundoora, Vic. 3083, *Australia*

AND P. NICKOLAS
University of Wollongong, Wollongong, N.S.W. 2500, *Australia*

(*Received* 29 *October* 1985)

1. *Introduction*

In this paper we prove a theorem which gives general conditions under which the free abelian topological group $F(Y)$ on a space Y can be embedded in the free abelian topological group $F(X)$ on a space X.

Roughly speaking, the theorem yields three classes of examples. Firstly, if Y is 'nice enough' and is a subspace of X, then $F(Y)$ can be embedded in $F(X)$. For example, if $Y = \mathbf{R}^n$, for any positive integer n, and X is any completely regular Hausdorff space containing $\mathbf{R}^n$, then $F(\mathbf{R}^n) \leqslant F(X)$, For $n = 1$ and $X = [0,1]$ this yields the main result of [**3**]. Secondly, if X is 'nice enough' and there is a continuous one-to-one mapping of Y into X, then $F(Y) \leqslant F(X)$. For example, if X is the Hilbert cube I^∞, then a necessary and sufficient condition for $F(Y)$ to be a subgroup of $F(X)$ is that Y is a submetrizable k_ω-space. Thirdly, if Y is 'nicely embedded' in X, then $F(Y) \leqslant F(X)$. For example, if $X = [0,1]^{n+1}$ and Y is a k_ω-space which is embedded in X in such a way that $Y \subseteq [0,1]^n \subseteq [0,1]^{n+1} = X$, then $F(Y) \leqslant F(X)$.

2. *Preliminaries*

We first record the necessary definitions and background results.

A Hausdorff topological space X is said to be a k_ω-space with k_ω-decomposition $X = \bigcup_n X_n$ if X_n is compact, $X_n \subseteq X_{n+1}$ for $n = 1, 2, 3, \ldots$ and X has the weak topology with respect to the sets X_n.

Definition. If X is a topological space with distinguished point e, the abelian topological group $F(X)$ is said to be the (Graev) *free abelian topological group on* X if

(a) the underlying group of $F(X)$ is the free abelian group with free basis $X\backslash\{e\}$ and identity e, and

(b) the topology of $F(X)$ is the finest topology on the underlying group which makes it into a topological group and induces the given topology on X.

If X is any completely regular space, then $F(X)$ exists, is unique, and is independent of the choice of e in X. Further, $F(X)$ is algebraically the free abelian group on $X\backslash\{e\}$. If X is also Hausdorff, then $F(X)$ is Hausdorff and has X as a closed subspace [**5**]. For k_ω-spaces, one can say rather more:

THEOREM A [4]. *Let $X = \bigcup X_n$ be any k_ω-space with distinguished point e. Then $F(X)$ is a k_ω-space and $F(X)$ has k_ω-decomposition $F(X) = \bigcup_n \mathrm{gp}_n(X_n)$, where $\mathrm{gp}_n(X_n)$ is the set of words of length not exceeding n in the subgroup generated by X_n.*

Remark. It is known [1] that every k_ω-topological group is a complete topological group.

Definition. Let $X = \bigcup X_n$ be a k_ω-space, and let $Y = \bigcup Y_n$ be a closed k_ω-subspace of $F(X)$. Then Y is said to be *regularly situated* with respect to X if for each natural number n there is an integer m such that $\mathrm{gp}(Y) \cap \mathrm{gp}_n(X_n) \subseteq \mathrm{gp}_m(Y_m)$.

THEOREM B [4]. *If X is a k_ω-space and Y is a closed subset of $F(X)$ such that $Y\backslash\{e\}$ is a free algebraic basis for* $\mathrm{gp}(Y)$, *and Y is regularly situated with respect to X, then* $\mathrm{gp}(Y)$ *is $F(Y)$.*

3. *Results*

THEOREM 1. *Let X be any completely regular Hausdorff space, $Y = \bigcup Y_n$ a k_ω-space, Γ a one-to-one continuous mapping of Y into X, and e any point of $\Gamma(Y_1)$. If, for each $n \in \mathbb{N}$, is a continuous function*

$$f_n: \Gamma((Y_n\backslash Y_{n-1}) \cup \partial_{Y_n}(Y_{n-1})) \to \Gamma(Y_n) \cup (X\backslash\Gamma(Y))$$

where $Y_0 = \varnothing$, such that

(i) $f_n(\Gamma(y)) = \Gamma(y)$, *for* $y \in \partial_{Y_{n+1}}(Y_n)$,

(ii) $f_n(\Gamma(y)) = e$, *for* $y \in \partial_{Y_n}(Y_{n-1})$,

then θ: $Y \to F(X)$ given by

$$\theta(y) = (n+1)\,\Gamma(y) + f_n(\Gamma(y)) \quad \textit{for} \quad y \in Y_n\backslash Y_{n-1}$$

extends to an embedding of $F(Y)$ in $F(X)$.

Proof. Initially assume that X is compact. We must show firstly that θ: $Y \to \theta(Y)$ is a homeomorphism. Let $y_1, y_2 \in Y$, $y_1 \neq y_2$. Suppose $\theta(y_1) = \theta(y_2)$. If $y_1, y_2 \in Y_n\backslash Y_{n-1}$, then $(n+1)\Gamma(y_1) + f_n(\Gamma(y_1)) = ((n+1)\Gamma(y_2) + f_n(\Gamma(y_2))$, and as $n \geqslant 1$, we have $\Gamma(y_1) = \Gamma(y_2)$, which implies that $y_1 = y_2$, a contradiction. Therefore, without loss of generality, $y_1 \in Y_n\backslash Y_{n-1}$ and $y_2 \in Y_{n_1}\backslash Y_{n_1-1}$, for some $n_1 \leqslant n-1$. But

$$\theta(y_2) = (n_1+1)\,\Gamma(y_2) + f_{n_1}(\Gamma(y_2)),$$

and
$$f_{n_1}(\Gamma(y_2)) \in \Gamma(Y_{n_1}) \cup (X\backslash\Gamma(Y)) \subseteq \Gamma(Y_{n-1}) \cup (X\backslash\Gamma(Y)),$$

so that $\theta(y_2) \neq \theta(y_1) = (n+1)\Gamma(y_1) + f_n(\Gamma(y_1))$. Hence θ is one-to-one.

To see that θ is continuous, observe firstly that for each $n \in \mathbb{N}$ and for all

$$y \in (Y_n\backslash Y_{n-1}) \cup \partial_{Y_n}(Y_{n-1}), \quad \theta(y) = (n+1)\,\Gamma(y) + f_n(\Gamma(y)).$$

As Y is a k_ω-space, it suffices to show that $\theta|Y_n$ is continuous for all n. We show this by induction, by observing that $\theta|Y_1$ is continuous and that if $\theta|Y_{n-1}$ is continuous, then as $\theta|(Y_n\backslash Y_{n-1}) \cup \partial_{Y_n}(Y_{n-1})$ is continuous, $\theta|Y_n$ is continuous, since

$$Y_n = Y_{n-1} \cup [(Y_n\backslash Y_{n-1}) \cup \partial_{Y_n}(Y_{n-1})]$$

and both Y_{n-1} and $(Y_n\backslash Y_{n-1}) \cup \partial_{Y_n}(Y_{n-1}) = \overline{(Y_n\backslash Y_{n-1})}$ are closed sets.

We now claim that $\theta(Y)$ is a closed subset of $F(X)$. As $\theta(Y) \cap F_n(X) = \theta(Y_n) \cap F_n(X)$, which is compact and hence closed, the k_ω-property of $F(X)$ implies that $\theta(Y)$ is closed in $F(X)$. Further the above equality then shows that $\theta(Y)$ is a k_ω-space with k_ω-decomposition $\bigcup\theta(Y_n)$.

As each $\theta|Y_n$ is a homeomorphism, it follows that $\theta: Y \to \theta(Y)$ is a homeomorphism.

The next step is to show that $\theta(Y)\backslash\{e\}$ is an algebraically free basis for the group it generates, and that $\theta(Y)$ is regularly situated with respect to X. Let w be any word in $\operatorname{gp}(\theta(Y))\backslash\{e\}$ with reduced representation in $\operatorname{gp}(\theta(Y))$

$$\begin{aligned} w &= m_1\theta(y_1) + m_2\theta(y_2) + \ldots + m_l\theta(y_l) \\ &= m_1[(n_1+1)\Gamma(y_1) + f_{n_1}(\Gamma(y_1))] + \ldots + m_l[(n_l+1)\Gamma(y_l) + f_{n_l}(\Gamma(y_l))]. \end{aligned} \tag{1}$$

The length of w with respect to $\theta(Y)$ is $\sum_{i=1}^{l}|m_i|$. Now as no $\Gamma(y_i)$ can cancel $\Gamma(y_j)$ for $j \neq i$, the length of w with respect to X is at least

$$\begin{aligned} &|m_1|(n_1+1) + \ldots |m_l|(n_l+1) - |m_1| - |m_2| - \ldots - |m_l| \\ &\geqslant 2|m_1| + \ldots + 2|m_l| - |m_1| - \ldots - |m_l| \\ &= |m_1| + \ldots + |m_l|. \end{aligned}$$

Thus the length of w with respect to X is greater than or equal to the length of w with respect to $\theta(Y)$; that is,

$$\operatorname{gp}(\theta(Y)) \cap \operatorname{gp}_n(X) \subseteq \operatorname{gp}_n(\theta(Y)). \tag{2}$$

From this we see that $\theta(Y)\backslash\{e\}$ is algebraically a free basis for $\operatorname{gp}(\theta(Y))$.

To prove that $\theta(Y)$ is regularly situated with respect to X, we shall extend (2) to

$$\operatorname{gp}(\theta(Y)) \cap \operatorname{gp}_n(X) \subseteq \operatorname{gp}_n(\theta(Y_n)). \tag{3}$$

To do this, consider a word w, as in (1), and suppose, without loss of generality, that

$$y_1, y_2, \ldots, y_s \in Y_{n_1}\backslash Y_{n_1-1} \quad \text{and} \quad y_{s+1}, \ldots, y_l, \in Y_{n_1-1}.$$

We claim that, after all possible cancellation, at least one of the following must appear in the reduced representation of w with respect to X:

$$m_1(n_1+1)\Gamma(y_1), m_1n_1\Gamma(y_1), m_2(n_1+1)\Gamma(y_2), m_2n_1\Gamma(y_2), \ldots, m_s(n_1+1)\Gamma(y_s), m_sn_1\Gamma(y_s).$$

In the word w consider the block

$$m_1[(n_1+1)\Gamma(y_1) + f_{n_1}(\Gamma(y_1))] + \ldots + m_s[(n_1+1)\Gamma(y_s) + f_{n_1}(\Gamma(y_s))],$$

where, without loss of generality, m_1 is greater than or equal to $m_2, \ldots, m_s$.

Step 1 is to observe that, since Γ is one-to-one, no $\Gamma(y_i)$, $i = 1, \ldots, s$, can be cancelled out by $\Gamma(y_j)$, for $j \neq i$.

Step 2 is to observe that no $f_{n_j}(\Gamma(y_j))$, $j > s$, can cancel out a $\Gamma(y_i)$, for $i = 1, \ldots, s$.

Step 3 is to consider the case when for all

$$i,j = 1, \ldots, s \quad \text{and} \quad i \neq j, f_{n_1}(\Gamma(y_i)) \neq f_{n_1}(\Gamma(y_j)).$$

It is readily seen using steps 1 and 2 that, since $m_1 \geqslant \max\{m_2, \ldots, m_s\}$, in the reduced representation of w, $m_1n_1\Gamma(y_1)$ must appear.

Step 4 is to consider the case when $f_{n_1}(\Gamma(y_i)) = f_{n_1}(\Gamma(y_j))$, for some $i, j \in (1, \ldots, s\}$, $i \neq j$. Then at most $s-1$ of $\Gamma(y_1), \ldots, \Gamma(y_s)$ can equal one of $f_{n_1}(\Gamma(y_1)), \ldots, f_{n_1}(\Gamma(y_s))$, and so for some $k \in \{1, \ldots, s\}$,

$$\Gamma(y_k) \neq f_{n_1}(\Gamma(y_r)), \quad \text{for} \quad r = 1, \ldots, s.$$

Hence the term $m_k(n_1+1)\Gamma(y_k)$ appears in the reduced representation of w. This completes the proof that $\theta(Y)$ is regularly situated with respect to X.

Thus we have proved the theorem for the case when X is compact.

It remains to consider the case when X is not compact. Here let $\beta: X \to \beta X$ be the embedding of X in its Stone–Čech compactification βX. Then β extends to a continuous one-to-one homomorphism $\beta: F(X) \to F(\beta X)$. As $\beta: X \to \beta X$ is an embedding, $\beta: \Gamma(Y) \to \beta(\Gamma(Y))$ is a homeomorphism. Defining $\theta: Y \to F(X)$ as earlier, we see that gp $(\theta(Y))$ is algebraically free on $\theta(Y)\backslash\{e\}$. Also applying the theorem as proved so far with X, Γ, f_n replaced, respectively, by βX, $\beta\Gamma$, βf_n, and with Y as before, we obtain a map $\theta': Y \to F(\beta X)$ which extends to an embedding of $F(Y)$ in $F(\beta X)$. Clearly $\theta' = \beta\theta$, and the fact that θ' extends to an embedding of $F(Y)$ in $F(\beta X)$ then implies that θ extends to an embedding of $F(Y)$ in $F(X)$. ∎

An important special case of Theorem 1 is when Y is a subspace of X and Γ is the natural embedding:

COROLLARY 1. *Let X be any completely regular Hausdorff space, $Y = \bigcup Y_n$ a k_ω-space which is a (not necessarily closed) subspace of X, and e any point of Y_1. If for each $n \in \mathbf{N}$ there is a continuous function*

$$f_n: Y_n\backslash Y_{n-1} \cup \partial_{Y_n}(Y_{n-1}) \to Y_n \cup X\backslash Y,$$

where $Y_0 = \varnothing$, such that

(i) *$f_n(y) = y$, for $y \in \partial_{Y_{n+1}}(Y_n)$,*

(ii) *$f_n(y) = e$, for $y \in \partial_{Y_n}(Y_{n-1})$,*

then $\theta: Y \to F(X)$ given by

$$\theta(y) = (n+1)y + f_n(y), \quad \textit{for} \quad y \in Y_n\backslash Y_{n-1}$$

extends to an embedding of $F(Y)$ in $F(X)$. ∎

As a consequence of the proof of Theorem 1, we obtain:

COROLLARY 2. *In the notation of the above theorem, if each f_n maps*

$$\Gamma((Y_n\backslash Y_{n-1}) \cup \partial_{Y_n}(Y_{n-1}))$$

into $\Gamma(Y_n)$ then gp $(\Gamma(Y))$ *has a subgroup topologically isomorphic to $F(Y)$.* ∎

Roughly speaking, Corollary 2 implies that if Y is a subspace of X and Y is 'nice enough', then gp (Y) contains $F(Y)$, irrespective of the space X. For example, this is the case when $Y = \mathbf{R}^n$.

COROLLARY 3. *Let X be any completely regular Hausdorff space and n any positive integer. If X has $\mathbf{R}^n$ as a subspace, then $F(X)$ has $F(\mathbf{R}^n)$ as a topological subgroup.*

Proof. $\mathbf{R}^n$ has k_ω-decomposition $\bigcup Y_n$, where $Y_n = \{x \in \mathbf{R}^n: |x| \leqslant n\}$, and the domain of f_n must then be $\{x \in \mathbf{R}^n: n \geqslant |x| \geqslant n-1\}$. We take e as the origin in $\mathbf{R}^n$ and define $f_n(x) = (|x| - n + 1)x$. Verification of the required properties of f_n is routine, and it then follows from Corollary 2 that gp$(\mathbf{R}^n)$ contains $F(\mathbf{R}^n)$. ∎

Example. $F([0, 1]^n)$ contains $F(\mathbf{R}^n)$ as a topological subgroup.

The case $n = 1$ of this example is the main result of [3].

COROLLARY 4. *Let X be any completely regular Hausdorff space, n any positive integer, and $\mathbf{R}^n$ a subspace of X. If Y is any closed subspace of $\mathbf{R}^n$, then $F(X)$ has $F(Y)$ as a topological subgroup.*

Proof. By Corollary 3, $F(X)$ contains $F(\mathbf{R}^n)$. As $F(\mathbf{R}^n)$ contains $F(Y)$ (for example, because Y is regularly situated with respect to $\mathbf{R}^n$), $F(X)$ contains $F(Y)$, from which the result follows. ▎

COROLLARY 5. *Let X be any completely regular Hausdorff space and $Z_1, Z_2, \ldots, Z_n, \ldots$ a countably infinite family of pairwise disjoint compact subspaces of X. If $Z = \coprod_n Z_n$ is the disjoint union of $Z_1, Z_2, \ldots$, then $F(X)$ has $F(Y)$ as a topological subgroup.*

Proof. Let Γ be the natural one-to-one continuous mapping of Z onto

$$\coprod_{n=1}^{\infty} Z_n \subseteq F(X).$$

Putting $Y_n = \coprod_{i=1}^{n} Z_i$, we see that $\partial_{Y_n}(Y_{n-1}) = \varnothing$, for all $n \geqslant 1$. Thus we can put each f_n equal to the identity mapping and the conditions of the theorem are satisfied. ▎

Example. Let X be any infinite completely regular Hausdorff space. Taking the Z_i in Corollary 5 to be distinct singleton sets, we see that $F(X)$ contains $F(Y)$, where Y is a countably infinite discrete space.

To date, our examples were all such that the mappings f_n were as in Corollary 2. We now consider a different situation. Suppose that X is any completely regular Hausdorff space, $Y = \bigcup Y_n$ is a k_ω-space, and Γ is a continuous one-to-one mapping of Y into X. Suppose further that each $\Gamma(Y_n)$ is 'nicely embedded' in the following sense: for each n, there is a positive integer $k = k(n)$ and a subspace T_n of X homeomorphic to $[0, 1]^k$ such that $\Gamma(Y_n) \subseteq T_n$. For convenience identify T_n with $[0, 1]^k$ and suppose, without loss of generality, that $e = (0, 0, \ldots, 0)$, for all k. Further, suppose that for each $y \in Y_n$, $\Gamma(y) = (y_1, \ldots, y_{k-1}, 0) \in [0, 1]^k$.

COROLLARY 6. *Under the above conditions, $F(X)$ has a subgroup topologically isomorphic to $F(Y)$.*

Proof. It suffices to show that Y has the mappings of Theorem 1. In what follows, note that Γ is a homeomorphism on each compact set Y_n.

Fix $n \in \mathrm{N}$. By Tietze's Theorem, there exists a continuous map

$$\Phi\colon \Gamma((Y_n \backslash Y_{n-1}) \cup \partial_{Y_n}(Y_{n-1})) \to I^{k-1}$$

such that for $\Gamma(y) = (y_1, \ldots, y_{k-1}, 0) \in \Gamma(\partial_{Y_{n+1}}(Y_n))$, $\Phi_n(\Gamma(y)) = (y_1, \ldots, y_{k-1})$, and for $\Gamma(y) = (y_1, \ldots, y_{k-1}, 0) \in \Gamma(\partial_{Y_n}(Y_{n-1}))$, $\Phi_n(\Gamma(y)) = (0, 0, \ldots, 0) \in I^{k-1}$. It is easily derived from Tietze's Theorem that if C is closed in a metric space Z, then there exists a continuous map $\delta\colon Z \to [0, 1]$ such that $C = \delta^{-1}(\{0\})$. Let

$$Z = \Gamma((Y_n \backslash Y_{n-1}) \cup \partial_{Y_n}(Y_{n-1}))$$

and

$$C = \Gamma(\partial_{Y_{n+1}}(Y_n) \cup \partial_{Y_n}(Y_{n-1})).$$

The required functions f_n are given by

$$f_n(\Gamma(y)) = (\Phi_n(\Gamma(y)), \delta(\Gamma(y))), \quad y \in (Y_n \backslash Y_{n-1}) \cup \partial_{Y_n}(Y_{n-1}).$$ ▎

As an immediate consequence of Corollary 6 we have

COROLLARY 7. *If Y is a k_ω-space which is a subspace of $[0, 1]^n$, then $F(Y)$ is topologically isomorphic to a subgroup of $F([0, 1]^{n+1})$.* ▎

For example if Y is any open subset of $[0, 1]^n$, then [5] Y is a k_ω-space and so $F([0, 1]^{n+1})$ contains $F(Y)$.

Since any separable metrizable space of dimension n is contained in $[0, 1]^{2n+1}$, we then obtain

COROLLARY 8. *If Y is a metrizable k_ω-space with* $\dim(Y) = n$, *then* $F(Y)$ *is topologically isomorphic to a subgroup of* $F([0, 1]^{2n+2})$.

Corollary 6 clearly remains true if $k = k(n)$ is $\aleph_0$, with the obvious notational changes.

We denote the Hilbert cube, the countably infinite product of unit intervals, by I^∞. Since any metrizable k_ω-space is separable and hence a subspace of I^∞, we obtain the following:

COROLLARY 9. *If Y is a metrizable k_ω-space, then $F(I^\infty)$ contains $F(Y)$.* ∎

In the theorem below, we shall give a characterization of the closed subgroups of $F(I^\infty)$, but first we need a lemma.

Recall that a topological space X is said to be *submetrizable* if it admits a continuous metric; that is, if there exists a metric on X which induces a topology no finer than the given topology.

LEMMA. *A k_ω-space $Y = \bigcup Y_n$ is submetrizable if and only if each Y_n is metrizable.*

Proof. If Y is submetrizable, then the compactness of each Y_n implies that it is metrizable.

Conversely, assume that each Y_n is metrizable. It suffices to show that there is a continuous one-to-one mapping of Y into I^∞, and for this it is enough to show that there exists a countable family of continuous maps of Y into $I = [0, 1]$ which separates points. As each Y_n is metrizable, there is a countable family of continuous maps of Y_n into I which separates points, and since Y is a k_ω-space, and hence normal, Tietze's Theorem shows that each map in this family extends to a continuous map of Y into I. Observing that each pair of points in Y lies in some Y_n, we see that the result follows. ∎

THEOREM 2. *Let Y be a completely regular Hausdorff space. Then $F(Y)$ is topologically isomorphic to a closed subgroup of $F(I^\infty)$ if and only if Y is a submetrizable k_ω-space.*

Proof. Firstly assume that $F(Y)$ is a closed subgroup of $F(I^\infty)$. Then, since I^∞ is compact, $F(I^\infty)$ is a k_ω-space and hence its closed subgroup $F(Y)$ is a k_ω-space. As I^∞ is compact metrizable, so too is each $F_n(I^\infty)$ metrizable. Hence, by the above Lemma, $F(I^\infty)$ is submetrizable. Thus the subspace Y of $F(I^\infty)$ is submetrizable.

Conversely, assume that Y is a submetrizable k_ω-space. By the proof of the above Lemma, there exists a continuous one-to-one mapping of Y into I^∞. Corollary 6 (with $k = k(n)$ replaced by $\aleph_0$) then implies that $F(Y)$ is topologically isomorphic to a subgroup of $F(I^\infty)$. ∎

Remark. Note that the conditions of the theorem do not demand that Y itself be metrizable. For example, $Y = F([0, 1])$ is not metrizable [6] but satisfies the conditions of the theorem.

Example. Let Y be a submetrizable k_ω-space. Then $F(F(Y))$ is topologically isomorphic to a closed subgroup of $F(I^\infty)$.

REFERENCES

[1] D. C. HUNT and S. A. MORRIS. Free subgroups of free topological groups. *Proc. Second Internat. Conf. Theory of Groups, Canberra 1973*, Lecture Notes in Math. vol. 372 (Springer-Verlag, 1974), 377–387.

[2] E. KATZ and S. A. MORRIS. On metrizable k_ω-spaces. *Math. Chronicle* **12** (1983), 119–122.

[3] E. KATZ, S. A. MORRIS and P. NICKOLAS. A free subgroup of the free abelian topological group on the unit interval. *Bull. London Math. Soc.* **14** (1982), 399–402.

[4] J. MACK, S. A. MORRIS and E. T. ORDMAN. Free topological groups and the projective dimension of a locally compact abelian group. *Proc. Amer. Math. Soc.* **40** (1973), 303–308.

[5] S. A. MORRIS. Free abelian topological groups. In *Categorical Topology*, Proc. Conference Toledo, Ohio, 1983 (Heldermann-Verlag, 1984), 375–391.

[6] S. A. MORRIS and H. B. THOMPSON. Metrizability of free topological groups. *Bull. Austral. Math. Soc.* **33** (1986), 103–112.