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A CHARACTERIZATION OF THE TOPOLOGICAL GROUP OF REAL NUMBERS

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It is shown that a non-discrete locally compact Hausdorff group has each of its proper closed subgroups finite (respectively, discrete) if and only if it is topologically isomorphic to the circle group (respectively, the circle group or the group of real numbers).

In Chapter 2 of [1] Armacost records various characterizations, in the class of locally compact abelian groups, of familiar groups. In particular Isiwata [3] and Robertson and Schreiber [6] proved that for G a non-discrete locally compact Hausdorff abelian group, every proper closed subgroup of G is discrete if and only if G is topologically isomorphic to R, the additive groups of real numbers with the usual topology, or T the compact circle group. We extend this result by showing that the word "abelian" can be omitted, so that this characterization is valid in the class of locally compact Hausdorff groups.

THEOREM. *Let G be a non-discrete locally compact Hausdorff topological group. Then every proper closed subgroup of G is discrete if and only if G is topologically isomorphic to T or R.*

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PROOF. If G is T or R then, by for example [1, Corollary 1.4], every proper closed subgroup is discrete.

Conversely, assume that G has the stated property. Then the component, $C(G)$, of the identity must be equal to G or $\{1\}$, where 1 denotes the identity of G. If $C(G) = \{1\}$, then G is totally disconnected and so, by [2, Theorem 7.7], G has a basis of compact open subgroups. Thus G has a non-trivial open discrete subgroup, which implies that G is discrete. This contradicts our assumptions on G

So G is a connected locally compact Hausdorff group. Therefore by the Iwasawa Structure Theorem [4, p.118], G has a maximal compact (connected) subgroup K and subgroups $R_1,\ldots,R_n$ each topologically isomorphic to R, such that every element $g \in G$ can be decomposed uniquely and continuously in the form $g = x_1x_2\ldots x_nk$, where each $x_i \in R_i$, and $k \in K$. As R is not discrete and K is not discrete unless trivial, we see that G is topologically isomorphic either to R or to the compact connected group, K.

If G is topologically isomorphic to K, then it has a closed normal subgroup F such that G/F is a non-trivial Lie group (see section 6.5.3 of [5].) So every element of G/F lies in a one-parameter subgroup. (See Remarks 2.2.15 and Corollary 4.3.5. of [5].) Observing that the property of having every proper closed subgroup discrete is preserved under quotient mappings, we see that G/F has this property, and so each one-parameter subgroup, being non-discrete, must equal G/F. So G/F is abelian. Corollary 1.4 of [1] then implies that G/F is topologically isomorphic to T. As F is discrete, this implies G is locally isomorphic to T, and hence by the Corollary to Theorem 8 of [4], G is topologically isomorphic to T.

COROLLARY. *Let G be a non-discrete locally compact Hausdorff topological group. Then every proper closed subgroup of G is finite if and only if G is topologically isomorphic to T.*

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[4] Sidney A. Morris, *Pontryagin Duality and the Structure of Locally Compact Abelian Groups* (Cambridge University Press, Cambridge, 1977).

[5] John F. Price, *Lie Groups and Compact Groups* (Cambridge University Press, 1977).

[6] L.C. Robertson and B.M. Schreiber, "The additive structure of integer groups and p-adic number fields", *Proc. Amer. Math. Soc.* 19 (1968) 1453-1456.

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