

FREE ABELIAN TOPOLOGICAL GROUPS

Sidney A. Morris

The notions of 'free topological group' and 'free abelian topological group' were introduced in 1941 by A.A. Markov [95], with full details appearing in his paper "On free topological groups" [96], which appeared in 1945. He began that paper as follows (but in Russian):

"The primary goal of the following investigation is to study the normality of topological groups, which arises in a natural way in the general theory of topological groups. Indeed as L. Pontryagin proved in an unpublished letter to A. Weil, every (Hausdorff) topological group is completely regular. Since next to complete regularity, normality is the most interesting property of separation of spaces, it is natural to ask whether every (Hausdorff) topological group is a normal space."

Markov answered the question in the negative by showing that: every completely regular Hausdorff space can be embedded as a closed subspace of a Hausdorff topological group - namely the Markov free topological group, $FM(X)$, on X . By letting X be any completely regular Hausdorff space which is not a normal space, we then have that $FM(X)$ is a non-normal topological group (since a closed subspace of a normal space is normal).

Markov's proof was about 50 pages long and relied on 48 lemmas. In 1948, M.I. Graev [47] proved Markov's result, that X completely regular Hausdorff implies $FM(X)$ is Hausdorff and X is closed in $FM(X)$, in 11 pages (with no lemmas!). While today there are other proofs of this fact which are much shorter than Graev's, Graev's approach remains of interest as it gives information about free topological groups, free abelian topological groups and free products of topological groups not obtained otherwise.