FREE ABELIAN TOPOLOGICAL GROUPS ON SPHERES

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§1. Introduction and Preliminaries

WE begin by recording the definition of free abelian topological group.

DEFINITION [5]. Let X be a completely regular Hausdorff space. Then the abelian topological group $F(X)$ is said to be the (Markov) *Free abelian topological group on* X if

(i) X is a subspace of $F(X)$,
(ii) X generates $F(X)$ algebraically, and
(iii) for every continuous mapping ϕ of X into any abelian topological group G there exists a continuous homomorphism Φ of $F(X)$ into G which agrees with ϕ on X.

It is well-known [**2, 5**] that $F(X)$ exists, is Hausdorff, and that $F(X)$ is algebraically a free abelian group having free basis X.

Nickolas [**7**] showed that, in the non-abelian case, the free topological group on any finite-dimensional compact metrizable space can be embedded (by a topological group isomorphism) in the free topological group on the closed unit interval I. Little progress, however, has been made on the analogous problem for free abelian topological groups. Indeed we conjecture that even $F(I^2)$ cannot be embedded in $F(I)$. (Note, however, that $F(I)$ does contain $F((0, 1))$, but the embedding is not an obvious one [3].)

In §2 we construct an embedding of $F(S^1)$ in $F(I)$. Of course $F(I)$ contains many homeomorphic copies of S^1, but we need one which is a free basis for the subgroup, G, that it generates, and we require, moreover, that the induced topology on G be the free topology.

This result is generalized in two ways in §3, where it is shown that for each positive integer n both $F(S^n)$ and $F((S^1)^n)$ can be embedded in $F(I^n)$, where S^n denotes the n-sphere. To do this we show firstly that if X, A and B are compact Hausdorff spaces and $F(X)$ is embedded in $F(A)$, then $F(X \times B)$ can be embedded in $F(A \times B)$. Using this we prove that if the embedding of $F(X)$ in $F(A)$ is suitably nice, then $F(\Sigma X)$ can be embedded in $F(\Sigma A)$, where Σ denotes suspension.

We record here some notation and results we shall use later. From the definition it is clear that the topology of the free abelian topological group $F(X)$ is the finest group topology on the underlying free group which will induce the given topology on X. In [**1**], Graev gives a construction, which we will outline below, of a group topology on the free abelian group on

X. (Graev actually constructs a topology on both the free group and the free abelian group on the set X. In the former case his topology is not the free topology [**6**]. In the latter case it is shown in [**8**] that it is the free topology, but we shall not use this fact.)

Let X be a completely regular Hausdorff space. Then the topology of X is determined by a family of pseudometrics $\{\rho_i: i \in I\}$, for some index set I. Each such ρ is extended to a pseudometric ρ_1 on the free abelian group F on X in the following manner. Let $w_1, w_2 \in F$. Consider a pair of (not necessarily reduced) representations of w_1 and w_2 having the same length; that is $w_1 = a_1 + a_2 + \cdots + a_k$, $w_2 = b_1 + b_2 + \cdots + b_k$, where $a_1, \ldots, a_k, b_1, \ldots, b_k \in X \cup (-X) \cup \{0\}$, with 0 denoting the identity of F. Then Graev defines $\rho_1(w_1, w_2)$ to be

$$\inf\left\{\sum_{j=1}^{k} \rho(a_j, b_j)\right\},$$

where the infimum is taken over all such pairs of representations of any lengths. Here, for $x_1, x_2 \in X$, we define

$$\rho(x_1, 0) = \rho(-x_1, 0) = 1,$$
$$\rho(-x_1, -x_2) = \rho(x_1, x_2) \quad \text{and}$$
$$\rho(x_1, -x_2) = 2.$$

The family of all such extended pseudometrices ρ, determines a Hausdortt group topology which clearly induces the given topology on X. In §3 we shall need the following lemma, which is a refinement for abelian groups of a result of Graev [**1**, proof of Theorem 1].

LEMMA 1. *In the above notation, let* $w_1, w_2 \in F$ *have reduced representations* $w_1 = x_1 + x_2 + \cdots + x_n$, $w_2 = y_1 + y_2 + \cdots + y_m$ *for* $x_1, \ldots, x_n$, $y_1, \ldots, y_m \in X \cup (-X)$. *Then there exists a pair of representations of* w_1 *and* w_2, $w_1 = a_1 + a_2 + \cdots + a_K$ *and* $w_2 = b_1 + b_2 + \cdots + b_k$, *such that*

(i) $k \leq n + m$,

(ii) $\rho_1(w_1, w_2) = \sum_{j=1}^{k} \rho(a_j, b_j)$ (*that is, the infimum is achieved*),

(iii) *for each* j, a_j *and* $b_j \in \{x_1, \ldots, x_n, -x_1, \ldots, -x_n, y_1, \ldots, y_m, -y_1, \ldots, -y_m, 0\}$ (This list may contain repetitions.)

In contrast with the situation for abstract groups it is *not* true that every subgroup of a free abelian topological group is a free abelian topological group. The following theorem, which is used in §2 and §3, gives a positive result in this direction. (While we state the result for compact spaces, an appropriate analogue holds for k_ω-spaces [**4**].) Firstly we need some notation. If Y is a subset of $F(X)$, the subgroup generated by Y is denoted by $gp(Y)$; $gp_n(Y)$ denotes the subset of $gp(Y)$ consisting of those elements of reduced length less than or equal to n with respect to Y.

DEFINITION. Let Y be a compact subset of $F(X)$. Then Y is said to be *regularly situated with respect to X* if for each positive integer p there exists a positive integer q such

$$gp_p(X) \cap gp(Y) \subseteq gp_q(Y) \quad \ldots\ldots\ldots\ldots (*)$$

To verify condition (*) it suffices to show:

$$\left.\begin{array}{l}\text{For each positive integer } n \text{ there exists a}\\ \text{positive integer } m \text{ such that whenever } w \in gp(Y)\\ \text{has reduced length } n \text{ with respect to } Y, \text{ its}\\ \text{reduced length with respect to } X \text{ is at least } m.\end{array}\right\} \quad \ldots\ldots\ldots (**)$$

THEOREM A[1]. *Let X and Y be compact Hausdorff spaces, with Y a subspace of $F(X)$. If Y is a free algebraic basis for $gp(Y)$ and Y is regularly situated with respect to X, then $gp(Y)$ is the free abelian topological group on Y.*

In general, for any compact Hausdorff space X, there exist many Hausdorff group topologies on the free abelian group on X which induce the given topology on X. Observe, however, that since $gp_n(X)$ is compact in all of these topologies, it inherits the same topology in each.

Finally, note that any compact subspace Y of the free abelian topological group on a compact Hausdorff space X lies in $gp_n(X)$ for some n [**1**].

§2. Embedding $F(S^1)$ in $F(I)$

Our copy of S^1 in $F(I)$ will consist of the union of two homeomorphic copies of I which coincide at the end points. The first copy of I is $A = \{[x/5]+[(x+2)/5]+[(x+4)/5]\colon x \in I\}$ where the operations of addition and division inside the brackets [] are taken in $\mathbb{R}$. So $x/5$, $(x+2)/5$, $(x+4)/5$, all belong to I. On the other hand the operation of addition outside the brackets refers to addition in the free abelian topological group $F(I)$. So $[x/5]+[(x+2)/5]+[(x+4)/5]$ is a word of reduced length three in $F(I)$. The second copy of I is $B = \{[x/5]+[(x^2+2)/5]+[(x^3+4)/5]\colon x \in I\}$. Of course $x^2 = x \cdot x$ and $x^3 = x \cdot x \cdot x$, where the multiplication is taken in $\mathbb{R}$.

Let $K = A \cup B$. Then K is homeomorphic to S^1. We shall show that the subgroup, $gp(K)$, of $F(I)$ generated by K is $F(S^1)$. By Theorem A it suffices to show that K is a free algebraic basis for $gp(K)$ and K is regularly situated with respect to I.

Let $w = x+(y+2)/5+(z+4)/5$ be any element of K. From the definition of K we see that $x \in [0, 1/5]$, $(y+2)/5 \in [2/5, 3/5]$, and $(z+4)/5 \in [4/5, 1]$. We shall represent w by the ordered triple (x, y, z). As the intervals involved are disjoint we see that if $(x, y, z) \in K$ then its first coordinate x [respectively, its second coordinate y] cannot equal the second [respectively, first] or third coordinate of any $(x_1, y_1, z_1) \in K$.

We now proceed to show that K is regularly situated with respect to I. We shall do this by verifying that condition (**) of §1 is satisfied.

Let w belong to $gp(K)$ have the form $w = a_1w_1 + a_2w_2 + \cdots + a_kw_k$ where each a_i is an integer, $w_i = (x_i, y_i, z_i) \in K$, and $w_i \neq w_j$, for $i \neq j$. Put $n = \sum_{i=1}^{k} |a_i|$, so n is the length of w with respect to K.

To measure the possible cancellation in $gp(K)$ we introduce the following notion.

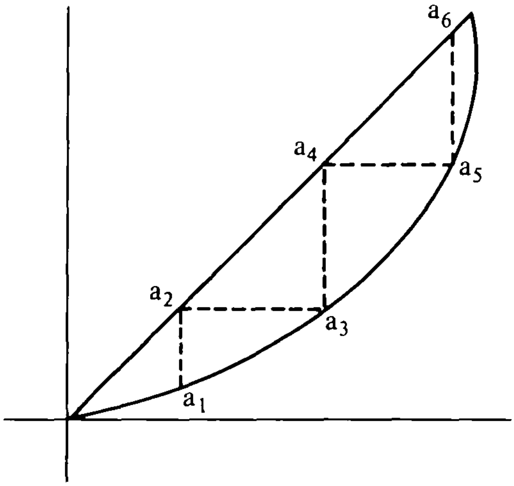
DEFINITION 1. An (x, y)-*sequence* from w is a maximal sequence $w_{i_1}, w_{i_2}, \ldots, w_{i_t}$ with $i_p \neq i_q$ for $p \neq q$ such that either

(a) $x_{i_1} = x_{i_2}$, $y_{i_2} = y_{i_3}$, $x_{i_3} = x_{i_4}$, $y_{i_4} = y_{i_5}$ and so on, or (b) $y_{i_1} = y_{i_2}$, $x_{i_2} = x_{i_3}$, $y_{i_3} = y_{i_4}$, $x_{i_4} = x_{i_5}$ and so on.

We define (y, z)-*sequences* similarly.

The following diagram may be helpful in picturing (x, y)-sequences.

We project K into the "(x, y)-plane". Then A is represented by the straight line and B by the curved line. An (x, y)-sequence zig-zags, as shown, from one side of the figure to the other. An (x, y)-sequence of type (a) starts at a_1, while one of type (b) starts at a_2.



Note that a sequence can be of length one.

LEMMA 2.

(i) *An element of K can occur in at most one (x, y)-sequence.*

(ii) *Distinct (x, y)-sequences do not have any point in common.*

(iii) *In* (i) *and* (ii) *above* (x, y)*-sequences can be replaced by* (y, z)*-sequences.*
(iv) *An* (x, y)*-sequence and a* (y, z)*-sequence can have at most two points in common.*

Proof. (i), (ii) and (iii) are obvious.

Suppose that an (x, y)-sequence and a (y, z)-sequence have a point in common. Then this point is of the form (x, x, x) or (x, x^2, x^3). Firstly consider the case where it has the form (x, x, x). Then a routine calculation shows that the first coordinate of every element in the (x, y)-sequence is x^{2n} for some integer n and the first coordinate of every element in the (y, z)-sequence, except perhaps $(x^{\frac{1}{2}}, x, x^{\frac{1}{3}})$, is $x^{2^m 3^k}$,. for non-zero integers m and k. Clearly then the (x, y)-sequence and the (y, z)-sequence have at most two points in common. The case where the common point has the form (x, x^2, x^3) is similarly handled. This completes the proof of the lemma.

We observe that if $w_{i_1}, w_{i_2}, \ldots, w_{i_l}$ as in Definition 1 is an (x, y)-sequence from $w = a_1 w_1 + a_2 w_2 + \cdots + a_k w_k$, and if (a) of Definition 1 applies then the reduced form of w with respect to I contains the symbols $a_{i_1} y_{i_1}$; if (b) applies, it contains $a_{i_1} x_{i_1}$. In addition the reduced form contains either $a_{i_l} x_{i_l}$ or $a_{i_l} y_{i_l}$, depending on the parity of l. Thus we obtain the following:

LEMMA 3. *Each* (x, y)*-sequence, even one of length one, contributes at least two letters to the reduced form of* w*, different from those contributed by any other* (x, y)*-sequence. The analogous result holds for* (y, z)*-sequences.*

PROPOSITION 1. *K is regularly situated with respect to I in $F(I)$.*

Proof. Consider the word w, as earlier. We wish to show that not too much cancellation occurs in it. The proof splits initially into two cases.

Case (a). Suppose that there are at least $n^{\frac{3}{4}}$ distinct w_i's. (Recall that $n = \sum_{i=1}^{k} |a_i|$.)

Subcase 1. Suppose that the number of (x, y)-sequences is $> n^{\frac{1}{4}}$. Then by Lemma 2 the reduced form of w with respect to I is of length at least $2n^{\frac{1}{4}}$.

Subcase 2. Suppose that there are at most $n^{\frac{1}{4}}$ (x, y)-sequences. As there are at least $n^{\frac{3}{4}}$ distinct w_i's and the (x, y)-sequences form a partition of the w_i's, there must be an (x, y)-sequence S of length at least $n^{\frac{1}{2}}$. By Lemma 2 each (y, z)-sequence has at most two points in common with S. But every element of S must lie in some (y, z)-sequence, so there are at least $\frac{1}{2}n^{\frac{1}{4}}$ (y, z)-sequences. By Lemma 3, then, the reduced form of w with respect to I is of length at least $n^{\frac{1}{4}}$.

Case (b). Suppose that there are strictly less than $n^{\frac{3}{4}}$ distinct w_i's. (So $k < n^{\frac{3}{4}}$.) Clearly, then, $|a_j| > n^{\frac{1}{4}}$, for some j.

Subcase 1. Suppose that there is an (x, y)-sequence in which either the first or the last symbol has $|a_i| \geqslant n^{\frac{1}{8}}$. Then by Lemma 3 the reduced length of w is $\geqslant n^{\frac{1}{8}}$.

Subcase 2. Suppose that every (x, y)-sequence has its first and last symbols satisfying $|a_i| < n^{\frac{1}{8}}$. Let S be an (x, y)-sequence $w_{i_1}, w_{i_2}, \ldots, w_{i_p}, \ldots, w_{i_l}$ with $|a_{i_p}| > n^{\frac{1}{4}}$.

Consider $w_{i_{p-1}}$. Then either its first coordinate or its second coordinate is equal to the corresponding coordinate of w_{i_p}. Then in the reduced form of w with respect to I, $n_i y_i$ or $n_i x_i$ appears, where $n_i \geqslant |a_{i_{p-1}} - a_{i_p}|$. Continuing this process we see that the reduced form of w with respect to I has at least

$$\begin{aligned} &|a_1 - a_2| + |a_2 - a_3| + \cdots |a_{i_{p-1}} - a_{i_p}| \\ &\geqslant |a_{i_1} - a_{i_p}| \\ &\geqslant |a_{i_p} - a_{i_1}| \\ &> n^{\frac{1}{4}} - n^{\frac{1}{8}} \quad \text{letters} \end{aligned}$$

Putting the cases together we see that condition (**) of §1 is satisfied and so K is regularly situated with respect to I in $F(I)$, completing the proof of the proposition.

Since a set Y satisfying condition (**) of §1 is clearly a free algebraic basis for $gp(Y)$ we have the following corollary.

COROLLARY 1. *K is a free algebraic basis for* $gp(K)$.

By Theorem A, Proposition 1, and Corollary 1, we obtain the main result of this section:

THEOREM 1. *$F(I)$ has a closed subgroup topologically isomorphic to* $F(S^1)$.

§3. Embedding F(S^n) in F(I^n)

DEFINITION 2. *Let the free abelian topological group $F(X)$ be topologically isomorphic to a subgroup of $F(A)$ for some spaces X and A, with ϕ: $F(X) \to F(A)$ being an embedding. For $x \in X$,* let *$\phi(x)$ have reduced representation with respect to A given by $\phi(x) = \varepsilon_1 a_1 + \cdots + \varepsilon_n a_n$ where $a_i \in A$ and $\varepsilon_i = \pm 1$ for each i. Then ϕ is said to be an orderly embedding if $\sum_{i=1}^{n} \varepsilon_i$ is independent of x.*

Remark. The embedding of $F(S^1)$ in $F(I)$ constructed in the proof of Theorem 1 is an orderly embedding.

THEOREM 2. *Let X, A and B be compact Hausdorff spaces. If there is an embedding ϕ of $F(X)$ in $F(A)$ then there exists an embedding Φ of $F(X\times B)$ in $F(A\times B)$. Further, if ϕ is an orderly embedding then Φ is an orderly embedding.*

Proof. Firstly let A and B be metric spaces, with the metrics on A and B being d_A and d_B, respectively. Define a metric ρ on $A\times B$ by $\rho((a_1, b_1), (a_2, b_2)) = d_A(a_1, a_2) + d_B(b_1, b_2)$ for $a_1 a_2 \in A$ and $b_1, b_2 \in B$. We also denote Graev's extension of ρ to $F(A\times B)$ by ρ, and, similarly, Graev's extension of d_A to $F(A)$ by d_A. As indicated in §1, d_A does not define the topology of $F(A)$, but it does, however, induce on each $gp_n(A)$ the same topology as does the free topology of $F(A)$. Thus d_A restricted to X defines the given topology of X. Define a metric ρ_1 on $X\times B$ by $\rho_1((x_1, b_1), (x_2, b_2)) = d_A(x_1, x_2) + d_B(b_1, b_2)$ for $x_1, x_2 \in X$ and $b_1, b_2 \in B$.

Let ϕ: $F(X) \to F(A)$ be the given embedding, where for $x\in X$, $\phi(x) = \varepsilon_1 a_1 + \cdots + \varepsilon_n a_n$ for $a_1, \ldots, a_n \in A$ and $\varepsilon_1, \ldots, \varepsilon_n = \pm 1$, and where $\varepsilon_1 a_1 + \cdots + \varepsilon_n a_n$ is in reduced form with respect to A. As X is compact we know the last paragraph of §1 that $\phi(X) \subseteq gp_N(A)$ for some N. Thus $n \leq N$ for each $x \in X$. We define Φ: $X\times B \to F(A\times B)$ by $\Phi(x, b) = \varepsilon_1(a_1, b) + \cdots + \varepsilon_n(a_n, b)$, $b\in B$. It is readily seen that Φ is one-to-one, that $\Phi(X\times B)$ is a free algebraic basis for the group it generates, and that $\Phi(X\times B)$ is also regularly situated with respect to $A\times B$. Thus to prove that the subgroup generated by $\Phi(X\times B)$ in $F(A\times B)$ is $F(X\times B)$, it suffices, by Theorem A, to show that Φ is continuous (and therefore a closed embedding, since $X\times B$ is compact).

Let (x_1, b_1) and (x_2, b_2) be elements of $X\times B$ such that $\rho_1((x_1, b_1), (x_2, b_2)) < \delta < 1$ for some $\delta > 0$. We shall show that $\rho(\Phi(x_1, b_1), \Phi(x_2, b_2)) < (2N+1)\delta$, so that Φ is seen to be continuous. For $i = 1$ and 2 let $x_i = \varepsilon_{i1} a_{i1} + \cdots + \varepsilon_{in_i} a_{in_i}$ be reduced representations with respect to A. Recall that the Graev extension of a metric (see §1) involves an infimum, and that this infimum is achieved for some representations $x_i = \eta_{i1} v_{i1} + \cdots + \eta_{im} v_{im}$, $v = 1$ and 2, of x_1 and x_2, where $v_{ij} \in A$ and $\eta_{ij} = \pm 1$, $j = 1, \ldots, m$ and $m \leq n_1 + n_2 \leq 2N$. As $\rho_1((x_1, b_1), (x_2, b_2)) < \delta < 1$ we can immediately deduce that $\eta_{1j} = \eta_{2j}$ for $j = 1, \ldots, m$ and that $v_{ij} \neq 0$. So $\rho_1((x_1, b_1), (x_2, b_2)) = \sum_{j=1}^{m} d_A(v_{1j}, v_{2j}) + d_B(b_1, b_2) < \delta$. Observe that for $i = 1, 2$, $\Phi(x_i, b_i) = \eta_{11}(v_{i1}, b_i) + \cdots + \eta_{1m}(v_{im}, b_i)$. (This is not quite as obvious as at first appears.) Using these representations of

$\Phi(x_1, b_1)$ and $\Phi(x_2, b_2)$, we see that

$$\begin{aligned}\rho(\Phi(x_1, b_1), \Phi(x_2, b_2)) &\leq \sum_{j=1}^{m} (d_A(v_{1j}, v_{2j}) + d_B(b_1, b_2)) \\ &\leq \sum_{j=1}^{m} d_A(v_{1j}, v_{2j}) + 2N\, d_B(b_1, b_2) \\ &\leq (2N+1)\delta.\end{aligned}$$

Hence Φ is continuous. So Φ: $X \times B \to F(A \times B)$ extends to an embedding Φ: $F(X \times B) \to F(A \times B)$. Clearly if ϕ is orderly then Φ is an orderly embedding.

If A and B now are arbitrary compact Hausdorff spaces, their topologies are determined by families $\{\rho_A\}$ and $\{\rho_B\}$ of continuous pseudometrics. By considering, as above, all possible pairs ρ_A, ρ_B in place of d_A, d_B, we obtain the desired result.

COROLLARY 2. *If for any positive integer n, $X_1, X_2, \ldots, X_n, A_1, A_2, \ldots, A_n$ are compact Hausdorff spaces and there are embeddings ϕ_i of $F(X_i)$ in $F(A_i)$, for each $i = 1, 2, \ldots, n$, then there is an embedding Φ of $F(X_1 \times X_2 \times \cdots \times X_n)$ in $F(A_1 \times A_2 \times \cdots \times A_n)$. Further, if each ϕ_i is an orderly embedding then Φ is an orderly embedding.*

COROLLARY 3. *There is an (orderly) embedding of $F((S^1)^n)$ in $F(I^n)$ for each n.*

THEOREM 3. *Let X and A be compact Hausdorff spaces. If there is an orderly embedding of $F(X)$ in $F(A)$, then there is an orderly embedding of $F(\Sigma X)$ in $F(\Sigma A)$, where Σ denotes the suspension.*

Proof. Consider the diagram

$$\begin{array}{ccc} X \times I & \xrightarrow{\Phi} & F(A \times I) \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_A} \\ \Sigma X & \xrightarrow[\Psi]{} & F(\Sigma A) \end{array}$$

in which Φ is the embedding given by Theorem 2 whose extension to an isomorphism on $F(X \times I)$ is orderly, where q_X is the canonical quotient mapping, and where q_A is the extension of the canonical quotient mapping of $A \times I$ onto ΣA. By the assumption of orderliness there is a unique (well-defined) one-to-one mapping Ψ (as shown) making the diagram commute. Then $\Psi \circ q_X = q_A \circ \Phi$, and since $q_A \circ \Phi$ is continuous and q_X is an identification, Ψ is continuous, and so is a closed embedding. It is easy to check that $\Psi(\Sigma X)$ is a set of free generators and is regularly situated in $F(\Sigma A)$, so that its extension Ψ: $F(\Sigma X) \to F(\Sigma A)$ is a closed embedding, which, further, is orderly.

COROLLARY 4. *For each positive integer n, there is an (orderly) embedding of $F(S^n)$ in $F(I^n)$,*

Proof. This follows immediately from Theorem 3 upon noting that ΣS^{n-1} is homomorphic to S^n, and ΣI^{n-1} is homeomorphic to I^n.

Remark. While Corollaries 3 and 4 are stated for Markov free abelian topological groups, the analogous results for Graev free abelian topological groups [**1**] can be easily deduced.

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