ARE FINITE TOPOLOGICAL SPACES WORTHY OF STUDY?

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I am writing a book on topology which I begin with a discussion of finite topological spaces. I have received some criticism over this from colleagues because they claim that finite topological spaces are not the slightest bit interesting. I challenge this claim by drawing attention to a little known but quite amusing result. Before doing so, however, let me go on the offensive by pointing out that many who advocate the alternative approach to topology via metric spaces and normed vector spaces usually spend much time examining various separable Banach spaces in great detail. They seem oblivious to the fact that all separable infinite-dimensional Banach spaces are homeomorphic! Therefore, as far as topology is concerned, they are studying the same object over and over and over again.

Now for the amusing result I promised. Let D denote the Davey topological space which consists of the three points 0, 1 and 2 with the only open sets being $\phi, \{0\}$, and $\{0,1,2\}$.

THEOREM *Every topological space (X,τ) is homeomorphic to a subspace of a product of homeomorphic copies of D.*

PROOF Let $J = K \cup X$ where $K = \{U : U \in \tau\}$.
For each $U \in K$, define a function $f_U : (X,\tau) \longrightarrow D$ by

$$f_U(y) = \begin{cases} 0, & y \in U \\ 1, & y \notin U \end{cases}.$$

For each $x \in X$, define $f_x : (X,\tau) \longrightarrow D$ by

$$f_x(y) = \begin{cases} 2, & \text{if } y = x \\ 1, & \text{if } y \neq x \end{cases}.$$

Let $e : (X,\tau) \longrightarrow \prod_{j\in J} D_j$, be the evaluation map, where each D_j is the Davey space; that is $e(y) = \prod_{j\in J} f_j(y)$, for each $y \in X$.

It is routine task to show that e is an embedding. (Probably the easiest way to do this is to apply the Embedding Lemma [1,p.116] - each f_j is continuous, $\{f_j : j \in X\}$ separates points of X, and $\{f_j : j \in K\}$ separates points and closed sets.) //

So every topological space can be obtained from a finite topological space using the operations of forming subspaces and products. So perhaps there is something of interest in finite spaces after all.

Of course I do not rest my case for teaching finite topological spaces first just on one amusing result. The main reason is that the approach provides a gentle introduction to abstract topological spaces in the same way that groups of small order provide a gentle introduction to group theory.

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REFERENCES

[1] John L. Kelly, *General Topology*, Von Nostrand, New York, 1969.

[2] Sidney A. Morris, *Topology Without Pain*, to appear.

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