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# A NOTE ON HOMEOMORPHIC MEASURES ON TOPOLOGICAL GROUPS

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## 1. Introduction

The classical von Neumann–Oxtoby–Ulam Theorem states the following:

Given non-atomic Borel probability measures $\mu, \lambda$ on $I^n$ such that

(1) $\mu(A)>0$, $\lambda(A)>0$ for all open $A\subset I^n$
(2) $\mu(\partial I^n)=\lambda(\partial I^n)=0$,

there exists a homeomorphism $h$ of $I^n$ onto itself fixing the boundary pointwise such that for any $\lambda$-measurable set $S$

$$\mu(h(S))=\lambda(S).$$

It is known that the above theorem remains valid if $I^n$ is replaced by any compact finite dimensional manifold [**2**], [**4**] or with $I^\infty$, the Hilbert cube, [**8**].

We shall say that a space $X$ has the *homeomorphic measure property* if the above theorem remains valid with $I^n$ replaced by $X$.

In this note we characterise those compact connected abelian metric groups having the homeomorphic measure property as precisely those which are locally connected. These are $T^a$, where $T$ is the circle group and $a$ is a non-negative integer or possibly $\aleph_0$.

Our proof depends on the following key result:

**Theorem A.** [**6**] *A countable product of finite dimensional compact manifolds has the homeomorphic measure property.*

## 2. Compact connected metrisable abelian groups

We shall regard $T$, the circle group, as the unit circle on the complex plane with complex multiplication as the group operation.

**Lemma 1.** *If $f$ is a continuous homomorphism of $G\cong T^m$, onto $H\cong T^n$, there exist subgroups $A$ and $B$ of $G$ such that $G=A\times B$, $A\cong T^n$ and $B\cong T^{m-n}$, $f(\{e\}\times B)=\{e\}$ and $f(A\times\{e\})=H$.*

**Proof.** Let $C$ be the kernel of $f$. Then $C=B\times F$ where $B\cong T^{m-n}$ and $F$ is a finite group. As $B\subseteq G$ and $B\cong T^{m-n}$, Section 25.31 in [3] implies $G=A\times B$ where $A\cong T^n$.

**Lemma 2.** *If $f$ is a continuous homomorphism of $G\cong T^n$ onto $H=H_1\times\cdots\times H_n$ where each $H_i\cong T$, then there exist subgroups $G_1,G_2,\ldots,G_n$ such that $G=G_1\times\cdots\times G_n$, $G_i\cong T$, and $f(G_i)=H_i$ for $i=1,2,\ldots,n$.*

**Proof.** Now $f^{-1}(H_1)$ is a subgroup of $T^n$ and so is $G_1\times F$ where $G_1\cong T^r$, $r\geqq 1$, and $F$ is a finite group. But then $f(G_1)$ is a connected subgroup of $H_1$ and so $f(G_1)=H_1$. As $G_1\cong T^r$, we must have $G=B\times G_1$ where $B=T^{n-r}$ and $f(B)=\{e\}\times H_2\times\cdots\times H_n\cong T^{n-1}$. So $n-r\geqq n-1$, which implies $r=1$.

So $G=B_1\times G_1$, $G_1\cong T$, $f(G_1)=H_1$ and $f(B_1)=\{e\}\times H_2\times\cdots\times H_n$. As above, there is a subgroup $G_2\subseteq B_1$ such that $f(G_2)=H_2$ and $G_2\cong T$. Continue inductively choosing $G_1,\ldots,G_n$.

So we have subgroups $G_1,\ldots,G_n$ of $G$ each isomorphic to $T$ and $G_i\cap G_j=\{e\}$ for $i\neq j$. Thus the group generated by $G_1,G_2,\ldots,G_n$ is $G_1\times\cdots\times G_n$ and is isomorphic to $T^n$. As $G_1\times\cdots\times G_n\subseteq G\cong T^n$ we must have $G=G_1\times\cdots\times G_n$ and the lemma is proved.

Observe that any continuous homomorphism $f$ of $T$ into $T$ is $f(e^{2\pi i\theta})=e^{2\pi ik\theta}$ where $k$ is a non-negative integer. We have demonstrated that any continuous homomorphism $f$ of $T^m$ onto $T^n$ is isomorphic to a canonical homomorphism $q$ which is a product of $n$ homomorphisms of $T$ into $T$. That is, there are isomorphisms $r$ and $r'$ of $T^m$ and $T^n$, respectively, such that $f=r'\circ q\circ r$.

Let $f$ be a canonical continuous homomorphism of $T^m$ onto $T^n$:

$$f(e^{2\pi i\theta_1},\ldots,e^{2\pi i\theta_n},\ldots,e^{2\pi i\theta_m})=(e^{2\theta ik_1\theta_1},\ldots,e^{2\pi ik_n\theta_n}).$$

If $m=n$, it is easily seen that $f$ is a $\prod_{i=1}^n k_i$ to one covering. In general, $f$ has connected kernel if and only if $k_i=1$ for each $i$ and if $A$ is a connected set of small enough diameter, then $f^{-1}(A)$ consists of $\prod_{i=1}^n k_i$ disjoint connected sets which are translates in $T^m$ of each other.

Suppose $G$ is a compact group and $g$ is a continuous homomorphism of $G$ onto $T^n$. Then $g$ preserves normalised Haar measure. This readily follows from the fact that $T^n$ is dyadically decomposable, that is, expressible as a disjoint union of sets which are translates of each other.

Suppose $G$ is a compact connected metrisable abelian group, then $G$ is an inverse limit space $\{G_i,p_i^{i+1}\}_{i\in\mathbb{N}}$ where the factor spaces $G_i=T^{n_i}$ for some positive integer $n_i$ and where the bonding maps $p_i^{i+1}$ are continuous surjective homomorphisms. (This well-known result follows easily for example from Corollary 1 of Theorem 14 of [5].) We shall let $\pi_i$ denote the projection of $G$ onto the factor $G_i$, so that $\pi_i=p_i^{i+1}\circ\pi_{i+1}$. If only a finite number of the bonding maps have disconnected kernel, then $G$ can be expressed as an inverse limit space $\{G_i',p_i'^{i+1}\}_{i\in\mathbb{N}}$ where all the bonding maps have connected kernels. Thus by Theorem 4.3 of [1], $G$ is locally connected.

**Lemma 3.** *Let $A$ be a closed arc contained in $G$. Let $\lambda$ and $\lambda_i$ be normalised Haar* *measures on G and $G_i$, respectively. If $p_i^{i+1}$ has disconnected kernel and $\pi_i(A) \neq G_i$ then*

$$\lambda(A) \leqq \lambda_{i+1}(\pi_{i+1}(A)) \leqq \tfrac{1}{2}\lambda_i(\pi_i(A)).$$

**Proof.** Let $\{A_k\}_k$ be a family of non-overlapping closed subarcs of $A$ such that $A = \bigcup_k A_k$. Suppose the lemma holds for each $A_k$ separately, then the lemma holds for $A$. Therefore, we may assume without loss of generality that $A$ is of as small a diameter as we choose, so that $q_i^{-1}(\pi_i(A))$, where $q_i = p_i^{i+1}$, is a disjoint union of closed connected sets $S_1, S_2, \ldots, S_m$, $m \geqq 2$, where each $S_j$ is a translate of $S_1$. Thus $\lambda_{i+1}(S_1) = \lambda_{i+1}(S_j)$ for each $j$ and since $p_i^{i+1}$ preserves Haar measure $\lambda_{i+1}(S_1) = (1/m)\lambda_j(\pi_i(A))$. By connectivity $\pi_{i+1}(A) \subseteq S_j$ for some $j$. This proves the lemma.

**Theorem 1.** *Let G be a compact connected metrisable abelian group. Then G has the homeomorphic measure property if and only if G is locally connected.*

**Proof.** Sufficiency follows from Theorem A. Suppose $G$ is not locally connected. As $G$ contains a one parameter subgroup, $G$ contains a closed arc $S$. Let $f$ be a homeomorphism of $I$ onto $S$ and define a Borel measure $\mu$ on $G$ *by* $\mu(A) = m(f^{-1}(A \cap S))$ where $m$ is linear Lebesgue measure. Define a Borel measure $\alpha$ on $G$ by $\alpha = \frac{1}{2}(\mu + \lambda)$. Then $\alpha$ is a locally positive non-atomic Borel probability measure on $G$ but cannot be homeomorphic to $\lambda$ because $\alpha(S) = \frac{1}{2}$ and by Lemma 3 $\lambda(h(S)) = 0$ for every homeomorphism $h$ of $G$ onto itself.

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