CHARACTERIZATION OF BASES OF SUBGROUPS OF FREE TOPOLOGICAL GROUPS

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ABSTRACT

It is shown here that if Y is a closed subspace of the free topological group $FM(X)$ on a k_ω-space X then $FM(X)$ has a closed subgroup topologically isomorphic to $FM(Y)$. Thus the problem of determining whether $FM(Y)$ can be embedded in $FM(X)$ is reduced to that of checking if $FM(X)$ contains a closed copy of Y.

As an extension of the above result it is shown that if $A_1, A_2, \ldots, A_n$ are (not necessarily distinct) closed subspaces of $FM(X)$, where X is a k_ω-space, then $FM(X)$ has a closed subgroup topologically isomorphic to $FM(A_1 \times A_2 \times \ldots \times A_n)$.

1. *Introduction and statement of the main theorem*

The main result of this paper is as follows.

THEOREM 1. *Let* $FM(X)$ *be the (Markov) free topological group on any* k_ω*-space* X *with at least two points. If* Y *is any closed subspace of* $FM(X)$, *then* $FM(X)$ *has a closed subgroup topologically isomorphic to* $FM(Y)$.

Since every Tychonoff space Y is closed in its free topological group $FM(Y)$ [**8**], Theorem 1 is optimal.

Nickolas [**11, 13**] proved a special case of Theorem 1, namely the case when X is the closed unit interval $[0, 1]$. His result appears as a corollary of his Kurosh subgroup theorem for topological groups which in turn relies on some heavy machinery for its proof. Our proof however is straightforward. Further, the Nickolas proof is not constructive while ours is.

Recall that a Hausdorff topological space is said to be a *k_ω-space* [**2, 3, 6**] if it is a countable union of compact spaces $X_1 \subseteq X_2 \subseteq \ldots \subseteq X_n \subseteq \ldots$ and has the weak topology with respect to these subspaces. As examples we have all countable CW-complexes, all connected locally compact Hausdorff groups, all open subspaces of compact metric space, and of course all compact Hausdorff spaces [**3**].

Now let X be any topological space and let $FM(X)$ be a topological group having X as a subspace. Then $FM(X)$ is said to be the *(Markov) free topological group on* X if

(i) X generates $FM(X)$ algebraically, and

(ii) every continuous map ϕ of X into any topological group G extends to a continuous homomorphism of $FM(X)$ into G.

References on free topological groups include [**1, 4, 5, 7, 8, 9, 10**]. We now record the results we shall need.

First, as every topological group is completely regular and every subspace of a completely regular space is completely regular, $FM(X)$ can exist only if X is

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completely regular. Indeed FM(X) does exist for all completely regular X and is unique (up to isomorphism) [**9**]. The underlying group of FM(X) is the free group on the set X [**9**]. If X is a k_ω-space with k_ω-decomposition $X = \bigcup X_n$, then FM(X) is a k_ω-space with k_ω-decomposition FM$(X) = \bigcup \text{gp}_n(X_n)$ where each $\text{gp}_n(X_n)$ is the compact subspace $(X_n \cup X_n^{-1} \cup \{e\})^n$ and e denotes the identity element [7].

If Y is a subspace of FM(X) then the subgroup gp(Y) generated by Y may or may not be the free topological group on Y. Of course, if Y is not a free algebraic basis for gp(Y), then gp(Y) is clearly not the free topological group on Y. However even if Y is a free algebraic basis it still need not be a free topological basis; that is, the topology on gp(Y) need not be that of FM(Y). One easy example of this is the following. Let Y be the open unit interval $(0, 1)$ and let X be the closed unit interval $[0, 1]$. Then the subgroup gp(Y) of FM(X) is not FM(Y) since it is not a closed subgroup of FM(X) (indeed it is a proper dense subgroup) while FM(Y) being a k_ω-group must be complete [5] and hence closed in any Hausdorff group which contains it. It is important to note that we have not shown that FM$[0, 1]$ does not contain FM$(0, 1)$, merely that gp(Y) is not FM$(0, 1)$. Indeed Nickolas [**13**] shows that FM$[0, 1]$ contains another copy Z of $(0, 1)$ such that gp(Z) is FM$(0, 1)$.

The following theorem gives a necessary and sufficient condition for a closed subspace Y of FM(X) to generate FM(Y).

THEOREM A [7]. *Let* FM(X) *be the (Markov) free topological group on a k_ω-space X with k_ω-decomposition $X = \bigcup X_n$. Let Y be a closed subspace of* FM(X) *with k_ω-decomposition $Y = \bigcup Y_n$. Then the subgroup,* gp(Y), *of* FM(X) *generated algebraically by Y is the free topological group* FM(Y) *on Y if and only if Y is a free algebraic basis for the free group,* gp(Y), *and for each positive integer n there exists a positive integer m such that* $\text{gp}(Y) \cap \text{gp}_n(X_n) \subseteq \text{gp}_m(Y_m)$.

DEFINITION. The topological space Y is said to be *enjoyably embedded* in the free topological group FM(X) if Y is a subspace of FM(X) and if for some $a \in X$ the subspace $Z = \{y^{-1}ay : y \in Y\}$ is homeomorphic to Y and satisfies gp(Z) = FM(Z).

In particular, if X is a k_ω-space and Y is closed in FM(X), then Z and FM(Z) are closed in FM(X). (This is seen by observing that if X is a k_ω-space then so is FM(X) and hence its closed subspace Y is also a k_ω-space. Therefore Z and FM(Z) are k_ω-spaces. As k_ω-groups are complete, FM(Z) is closed in FM(X). Finally note that Z is closed in FM(Z).)

2. *Proof of the main theorem and some corollaries*

A step in the proof of Theorem 1 is the following.

THEOREM 2. *Let $X = \bigcup X_n$ be a k_ω-space and let* FM(X) *be the free topological group on X. If Y is a closed subspace of* FM(X) *such that for some $a \in X$,* $\text{gp}(Y) \subseteq \text{gp}(X \setminus \{a\})$, *then Y is enjoyably embedded in* FM(Y).

Proof. Without loss of generality, let $a \in X_1$. Let Z be the subspace $\{y^{-1}ay : y \in Y\}$ of FM(X). Put $Y_n = Y \cap \text{gp}_n(X_n)$. As Y is closed, each Y_n is compact. Now let ϕ be the continuous map : $Y \to$ FM(X) given by $\phi(y) = y^{-1}ay$, for all $y \in Y$. Then $\phi(Y_n)$ is compact, for each n.

As $\mathrm{gp}(Y) \subseteq \mathrm{gp}(X \setminus \{a\})$, it follows that $Z \cap \mathrm{gp}_n(X_n) = \phi(Y_n) \cap \mathrm{gp}_n(X_n)$ and therefore is compact for each n. As $\mathrm{FM}(X)$ has k_ω-decomposition $\mathrm{FM}(X) = \bigcup \mathrm{gp}_n(X_n)$, we then have that Z is closed in $\mathrm{FM}(X)$.

Let $w = (y_1^{-1} a y_1)^{\varepsilon_1} (y_2^{-1} a y_2)^{\varepsilon_2} \ldots (y_m^{-1} a y_m)^{\varepsilon_m} \in \mathrm{gp}(Z)$, where each $\varepsilon_i = \pm 1$, $y_i = y_{i+1}$ implies that $\varepsilon_i = \varepsilon_{i+1}$, and $y_i \in Y$. Clearly no a can be cancelled out. Thus $w \neq e$, the identity element, and so Z is a free algebraic basis for the free group $\mathrm{gp}(Z)$.

To prove that $\mathrm{gp}(Z)$ is $\mathrm{FM}(Z)$, in view of Theorem A it suffices to show that

$$\mathrm{gp}(Z) \cap \mathrm{gp}_n(X_n) = \mathrm{gp}_n(\phi(Y_n)) \cap \mathrm{gp}_n(X_n). \tag{1}$$

Of course the right hand side of (1) is obviously contained in the left hand side. Now let $w \in \mathrm{gp}(Z) \cap \mathrm{gp}_n(X_n)$. As no a can be cancelled out, $m \leqslant n$. Thus $w \in \mathrm{gp}_n(\phi(Y))$. Let y_i, from the word w, have reduced representation

$$y_i = x_{i1}^{\eta_1} x_{i2}^{\eta_2} \ldots x_{ik}^{\eta_k}$$

with respect to X, where each $x_{ij} \in X$ and $\eta_j = \pm 1$. (Of course k is also dependent on i.) Because of the special form of the word w, each x_{ij} appears at least twice in the reduced form of w with respect to X. As $w \in \mathrm{gp}_n(X_n)$, each $x_i \in X_n$ and $k \leqslant n$, it follows that $y_i \in Y_n$. Hence $w \in \mathrm{gp}_n(\phi(Y_n))$, as required. So (1) is proved, and thus $\mathrm{gp}(Z)$ is $\mathrm{FM}(Z)$.

Finally we show that Z is homeomorphic to Y. Observe that Z is a k_ω-space with k_ω-decomposition $Z = \bigcup (Z \cap \mathrm{gp}_n(X_n))$. As ϕ is one-to-one, it suffices to show that $Z = \bigcup \phi(Y_n)$ is also a k_ω-decomposition. This in turn follows from the observations that

$$\phi(Y_n) \supseteq Z \cap \mathrm{gp}_n(X_n)$$

and

$$\phi(Y_n) \subseteq Z \cap \mathrm{gp}_{2n+1}(X_n) \subseteq Z \cap \mathrm{gp}_{2n+1}(X_{2n+1}).$$

The proof is now complete.

Corollary 1. *Let* $\mathrm{FM}(X)$ *be the free topological group on a non-compact* k_ω*-space* $X = \bigcup X_n$. *If* Y *is a compact subspace of* $\mathrm{FM}(X)$ *then* Y *is enjoyably embedded in* $\mathrm{FM}(X)$.

Proof. $\mathrm{FM}(X)$ has k_ω-decomposition $\mathrm{FM}(X) = \bigcup \mathrm{gp}_n(X_n)$. As Y is compact, $Y \subseteq \mathrm{gp}_n(X_n)$, for some n. Thus for any $a \in X \setminus X_n$, $Y \subseteq \mathrm{gp}(X \setminus \{a\})$, and so the conditions of Theorem 2 are satisfied.

Corollary 2. *Let* X *be a* k_ω*-space which contains a proper closed homeomorphic copy of itself. If* Y *is a closed subspace of* $\mathrm{FM}(X)$ *then* $\mathrm{FM}(X)$ *has a closed subgroup topologically isomorphic to* $\mathrm{FM}(Y)$.

Proof. Let the closed homeomorphic copy of X be X'. Then by Theorem A, $\mathrm{gp}(X') = \mathrm{FM}(X')$. As $\mathrm{FM}(X')$ is topologically isomorphic to $\mathrm{FM}(X)$, $\mathrm{FM}(X')$ contains a closed copy Y' of Y. Now Y' is closed in $\mathrm{FM}(X)$, since $\mathrm{FM}(X')$ is closed in $\mathrm{FM}(X)$ and Y' with respect to $\mathrm{FM}(X)$ satisfies the conditions of Theorem 2. From this the result follows.

COROLLARY 3 (Nickolas [13]). *If Y is any closed subspace of* FM $[0, 1]$, *then* FM $[0, 1]$ *has a closed subgroup topologically isomorphic to* FM (Y).

Proof. This is a special case of Corollary 2.

COROLLARY 4. *Let M be any second countable topological n-manifold with boundary. If Y is a closed subspace of* FM (M) *then* FM (M) *has a closed subgroup topologically isomorphic to* FM (Y).

Proof. Note that a second countable topological manifold with boundary is a locally compact σ-compact Hausdorff space and thus is a k_ω-space [3]. Further every n-manifold with boundary contains a closed copy of itself. (Let m be in the boundary of M. Let $U = \{(x_1, x_2, \ldots, x_n) = x \mid x_1 \geqslant 0\} \subseteq \mathbb{R}^n$ be homeomorphic to a neighbourhood of m such that under the homeomorphism, m corresponds to $(0, 0, 0, \ldots, 0)$. Then we can map $V = \{x \mid x \in U \text{ and } \|x\| \leqslant 1\}$ homeomorphically onto $V' = \{x \mid x \in V \text{ and } \frac{1}{2} \leqslant \|x\| \leqslant 1\}$ such that each $x \in V$ satisfying $\|x\| = 1$ is mapped onto itself. This map can be extended to a map of U into itself by mapping points outside V identically. This map induces a homeomorphism from M onto a proper closed subset of M.) Now we can apply Corollary 2 to obtain the required result.

We now proceed to prove Theorem 1.

Proof of Theorem 1. It is shown in [12] that FM (X) has a closed subgroup topologically isomorphic to FM $(X \times X)$. (This subgroup is actually the one generated by $\{xyx \mid x, y \in X\}$.) Now for any $x_0 \in X$, $X \times \{x_0\}$ is a closed subspace of $X \times X$ which, by Theorem A, generates in FM $(X \times X)$ a closed subgroup topologically isomorphic to FM (X). So this subgroup contains a closed homeomorphic copy of Y, which implies that for any $x_1, x_2 \in X$ with $x_2 \neq x_0$, $\mathrm{gp}(X \times X \setminus \{(x_1, x_2)\})$ contains a closed copy of Y. Thus by Theorem 2, FM $(X \times X)$ has a closed subgroup topologically isomorphic to FM (Y). Since FM (X) contains a closed copy of FM $(X \times X)$, it contains a closed subgroup topologically isomorphic to FM (Y).

COROLLARY 5. *If X is any k_ω-space with at least two points, then* FM (X) *has a closed subgroup topologically isomorphic to* FM $(\mathrm{FM}(X))$.

In the proof of Theorem 1 we used the result that FM (X) contains a copy of FM $(X \times X)$. We can now state a theorem which includes this and Theorem 1.

THEOREM 3. *Let $A_1, A_2, \ldots, A_n$ be (not necessarily distinct) closed subspaces of* FM (X), *where X is a k_ω-space with at least two points. Then* FM (X) *has a closed subgroup topologically isomorphic to* FM $(A_1 \times A_2 \times \ldots \times A_n)$.

Proof. Clearly it suffices to prove this for the case when $n = 2$. By Corollary 5, FM (X) contains FM $(\mathrm{FM}(X))$. By Nickolas [12], FM $(\mathrm{FM}(X))$ contains FM $(\mathrm{FM}(X) \times \mathrm{FM}(X))$; and Theorem A implies that FM $(\mathrm{FM}(X) \times \mathrm{FM}(X))$ contains FM $(A_1 \times A_2)$. As all of the above containments are closed, FM (X) has a closed subgroup topologically isomorphic to FM $(A_1 \times A_2)$.

3. *Sequential fans*

Let S denote the subspace of $\mathbb{R}$ consisting of the points 0 and $1/m$, for $m = 1, 2, \ldots$. Let T denote the topological space which is the one-point union of a countably infinite number of disjoint copies of S formed by identifying the copies of 0. This space is called the *sequential fan* and is a non-metrizable k_ω-space [**2**].

THEOREM 4. *Let* FM (X) *be the free topological group on any Hausdorff space* X *which has a non-trivial convergent sequence. Then* FM (X) *has a closed subgroup topologically isomorphic to* FM (T), *where* T *is the sequential fan.*

Proof. Let (x_n), where $x_n \neq x_0$, be a sequence in X converging to the point x_0 of X. Let K be the subspace of FM (X) consisting of x_0 and all the points x_n. Then K is compact and so, by Theorem 1.10 of [**10**], gp (K) is FM (K).

Let $y_n = x_0^{-1}x_n$ and let L_1 be the compact set consisting of all the y_n and the identity element e. Let $L_m = \{(y_n)^m\}_{n=1}^{\infty} \cup \{e\}$ and put $L = \bigcup_{m=1}^{\infty} L_m$.

As $L \cap \mathrm{gp}_n(K) = L_n \cap \mathrm{gp}_n(K)$, we see that L is closed in the k_ω-space FM (K) and is a sequential fan. By Theorem 1, then, FM (K) contains a closed subgroup topologically isomorphic to FM (L) = FM (T). As FM (X) contains FM (K) we have the required result.

As an immediate result of Theorem 4 we obtain Graev's result.

COROLLARY 6 [**4**]. *Let* FM (X) *be the free topological group on any non-discrete topological space* X. *Then* FM (X) *is not metrizable.*

Proof. This follows immediately from the fact that if X is non-discrete and metrizable then FM (X) has a subspace which is non-metrizable, namely the sequential fan.

REMARK. Let $X = \bigcup X_n$ be a non-discrete k_ω-space, where each X_n is metrizable. Then Franklin and Thomas [**2**] show that the non-metrizable group FM (X) contains a sequential fan. However our corollary below shows more.

COROLLARY 7. *Let* $X = \bigcup X_n$ *be a non-discrete* k_ω*-space where each* X_n *is metrizable. Then* FM (X) *has a closed subgroup topologically isomorphic to* FM (T), *where* T *is a sequential fan.*

4. *A non-*k_ω *result*

We now extend Theorem 2 by removing the k_ω-restriction on X, but we have to require that Y be compact.

THEOREM 5. *Let* Y *be a compact subspace of* FM (X), *where* X *is a completely regular Hausdorff space. If there is some* $a \in X$ *such that* $Y \subseteq \mathrm{gp}(X \setminus \{a\})$, *then* FM (X) *contains a closed subgroup topologically isomorphic to* FM (Y).

Proof. Let βX be the Stone–Čech compactification of X, so that βX is a compact Hausdorff space. The natural embedding of X in βX induces a continuous one-to-one homomorphism β of FM(X) into FM(βX). Let Z be the subspace $\phi(Y)$ of FM(X), where $\phi(y) = y^{-1}ay$, for all $y \in Y$. As Y is compact and ϕ is a continuous one-to-one map, Z is homeomorphic to Y. Now $Z' = \beta(Z)$ is a compact subspace of FM(βX) homeomorphic to Z. By Theorem A, gp(Z') is FM(Z'); and so the map β is a one-to-one continuous homomorphism of gp(Z) onto FM(Z'). The topology on gp(Z) must therefore be finer than that of FM(Z'), though the topology on Z is the same as that on Z'. But the topology of FM(Z') is the finest group topology on the underlying group which induces the given topology on Z' [**8**]. It follows that gp(Z) is FM(Z). Thus FM(Z) has a closed subgroup FM(Z) which is topologically isomorphic to FM(Y).

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