ON METRIZABLE k_ω-SPACES

Eli Katz and Sidney A. Morris

(received 13 October, 1981)

Abstract. In recent years k_ω-spaces have played a critical role in the study of topological groups [5, 6, 7, 8, 9, 11, 12]. (A Hausdorff space is said to be a k_ω-space if it is a countable union of compact spaces and has the weak topology with respect to these spaces.) The class of k_ω-spaces is known to be wide enough to include any countable *CW*-complexes and all locally compact σ-compact Hausdorff spaces, but restrictive enough that k_ω-condition makes it possible to handle topological problems by purely computational means.

Despite the widespread interest in k_ω-spaces and their use [1, 3, 4, 8, 10, 13] an easily proved but very pleasant result seems to have been overlooked. This result is that any open subspace of a compact metric space is a k_ω-space. Indeed metrizable k_ω-spaces can be characterised as open subspaces of metrizable compact spaces.

Theorem 1. *An open subspace of a compact metric space is a* k_ω*-space.*

Proof. Let Y be an open subspace of the compact metric space X. Then X is separable and so has a countable dense subset S. Let B be the set of all closed balls having centre in S and rational radius. Clearly B is a countable family of compact sets. We claim that Y is a union of members of B and hence is σ-compact.

Let $y \in Y$. Then Y is an open set containing y and so contains an open ball about y of some rational radius r, say. Then the open ball D about y of radius $r/3$ must contain some

Math. Chronicle 12(1983) 119-122.

element, s , of S . Now consider the closed ball B_1 with centre s and radius $r/3$. Clearly $B_1 \in B$, $y \in B_1$ and $B_1 \subseteq Y$. Thus Y is a union of members of B , as required.

So Y is σ-compact and, as it is an open subspace of a compact metric space, also locally compact Hausdorff. Thus Y is a k_ω-space ([4], 10).

Theorem 2. *Any metric k_ω-space can be embedded as an open subspace of a compact metric space.*

Proof. Let X be a metric k_ω-space and Y its one-point compactification. Being a metric k_ω-space, X is second countable ([4],19) and locally compact Hausdorff ([4], 21). Thus by Theorem 8.6 of [2], Y is metrizable. Of course X is an open subspace of Y .

Corollary. *A topological space is a metrizable k_ω-space if and only if it can be embedded as an open subspace of a metrizable compact space.*

Corollary. *Any open subspace of a metrizable k_ω-space is a k_ω-space.*

Examples : (i) $(0,1)$ is an open subspace of the compact metric space $[0,1]$ and is therefore a k_ω-space.

(ii) If X is any compact metric space and $x_1, \ldots, x_n$ are in X , then $X \setminus \{x_1, \ldots, x_n\}$, with the subspace topology is a k_ω-space.

(iii) Let X be an uncountable discrete space and Y its one-point compactification. Then X is not a k_ω-space but it is an open subspace of the compact Hausdorff space Y . Thus the metrizability condition cannot be dropped from Theorem 1.

REFERENCES

1. R. Brown and J.P.L. Hardy, *Subgroups of free topological groups and free topological products of topological groups*, J. London Math. Soc. (2) 10(1975), 431-440.

2. James Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.

3. Stanley P. Franklin and Barbara V. Smith Thomas, *On the metrizability of k_ω-spaces*, Pacific J. Math. 72(1977), 399-402.

4. Stanley P. Franklin and Barbara V. Smith Thomas, *A survey of k_ω-spaces*, Topology Proc. 2(1977), 111-124.

5. E. Katz, *Free products in the category of k_ω-groups*, Pacific J. Math. 59(1975), 493-495.

6. Elyahu Katz and Sidney A. Morris, *Free products of topological groups with amalgamation*, submitted.

7. Eli Katz and Sidney A. Morris, *Free products of k_ω-topological groups with normal amalgamation*, submitted.

8. W.F. La Martin, *On the foundations of k-group theory*, Dissertationes Math. (Rozprawy Mat.) 146(1977), 1-36.

9. John Mack, Sidney A. Morris and Edward T. Ordman, *Free topological groups and the projective dimension of a locally compact abelian group*, Proc. Amer. Math. Soc. 40(1973), 303-308.

10. E. Michael, *Bi-quotient maps and cartesian products of quotient maps*, Ann. Inst. Fourier (Grenoble) 18(1967), 278-302.

11. Peter Nickolas, *A Kurosh subgroup theorem for topological groups*, Proc. London Math. Soc. (3) 42(1981), 461-477.

12. E.T. Ordman, *Free products of topological groups which are k_ω-spaces*, Trans. Amer. Math. Soc. 19(1974), 61-73.

13. N.E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. 14(1967), 133-152.

Cleveland State University
Cleveland, Ohio, 44115,
U.S.A.

La Trobe University,
Bundoora, Vic. 3083,
AUSTRALIA