A FREE SUBGROUP OF THE FREE ABELIAN TOPOLOGICAL GROUP ON THE UNIT INTERVAL

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§1. *Introduction*

We prove that the free abelian topological group, $FA[0, 1]$, on the closed interval $[0, 1]$ has a closed subgroup topologically isomorphic to $FA(0, 1)$, the free abelian topological group on the open interval $(0, 1)$.

The analogue of this result for the free (non-abelian) topological group $F[0, 1]$, was proved by Nickolas [**4**], but his techniques rely heavily on non-commutativity and cannot be used here. (Note that in neither the abelian nor the non-abelian case does the obvious copy of $(0, 1)$ generate the desired subgroup. Moreover, any copy which does must be closed in $FA[0, 1]$ or $F[0, 1]$, respectively (see [**1, 3**]).) Furthermore, Nickolas's proof does not easily yield an explicit embedding of $F(0, 1)$ in $F[0, 1]$. Our proof, on the other hand, is constructive and thus does yield an explicit embedding of $FA(0, 1)$ in $FA[0, 1]$—and indeed the analogous construction explicitly embeds $F(0, 1)$ in $F[0, 1]$.

§2. *The main result*

We first record the necessary definitions and background results. These results are stated in the form we need rather than in their finest versions.

A Hausdorff topological space X is said to be a *k_ω-space* with *k_ω-decomposition* $X = \bigcup_n X_n$ if X_n is compact, $X_n \subseteq X_{n+1}$ for $n = 1, 2, 3, \ldots$ and X has the weak topology with respect to the sets X_n. For example, $(-1, 1)$ has k_ω-decomposition $(-1, 1) = \bigcup_n [-1+n^{-1}, 1-n^{-1}]$. Of course every compact Hausdorff space is trivially a k_ω-space.

DEFINITION. If X is a topological space with distinguished point e the abelian topological group $FA(X)$ is said to be the (Graev) *free abelian topological group on X* if

(a) the underlying group of $FA(X)$ is the free abelian group with free basis $X \setminus \{e\}$ and identity e, and

(b) the topology of $FA(X)$ is the finest topology on the underlying group which makes it into a topological group and induces the given topology on X.

THEOREM A [**2**]. *Let $X = \bigcup X_n$ be any k_ω-space with distinguished point e. Then $FA(X)$ exists, is unique (up to isomorphism), is independent of the choice of e in X, and has k_ω-decomposition $FA(X) = \bigcup_n gp_n(X_n)$, where $gp_n(X_n)$ is the set of words of length not exceeding n in the subgroup generated by X_n.*

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THEOREM B [2]. *Let* $X = \bigcup_n X_n$ *be a* k_ω*-space with distinguished point* e. *Let* $Y \subset FA(X)$ *be a subset containing* e *such that* $Y \setminus \{e\}$ *is a free algebraic basis for the subgroup,* $gp(Y)$, *that it generates. Suppose* $Y_1, Y_2, \ldots$ *is a sequence of compact subsets of* Y *such that* $Y = \bigcup Y_n$ *is a* k_ω*-decomposition of* Y *inducing the same topology on* Y *that* Y *inherits as a subset of* $FA(X)$. *If for each natural number* n *there is an integer* m *such that* $gp(Y) \cap gp_n(X_n) \subseteq gp_m(Y_m)$, *then* $gp(Y)$ *is* $FA(Y)$, *and both* $gp(Y)$ *and* Y *are closed subsets of* $FA(X)$.

REMARK. If in the above definition the word "abelian" is everywhere deleted then we have the definition of the *free topological group* $F(X)$ on the space X. The above theorems remain valid if $FA(X)$ and $FA(Y)$ are replaced everywhere by $F(X)$ and $F(Y)$, respectively.

We can now state the main result.

THEOREM 1. $FA(0,1)$ *is topologically isomorphic to a closed subgroup of* $FA[0,1]$.

Proof. We shall show that $FA(-1,1)$ is topologically isomorphic to a closed subgroup of $FA[-1,1]$. We let 0 be the distinguished point of $[-1,1]$.

We now define a map $\phi : (-1,1) \to FA[-1,1]$. First we put

$$f_n(x) = \begin{cases} (n+1)^2 x - n(n+1), & \text{if } x \in \left[\dfrac{n}{n+1}, \dfrac{n+1}{n+2}\right] \\ (n+1)^2 x + n(n+1), & \text{if } x \in \left[-\dfrac{n+1}{n+2}, -\dfrac{n}{n+1}\right] \end{cases}$$

for each non-negative integer n. Then we set

$$\phi(x) = (n+1)x + f_n(x) \quad \text{for } x \in \left[\frac{n}{n+1}, \frac{n+1}{n+2}\right] \cup \left[-\frac{n+1}{n+2}, -\frac{n}{n+1}\right],$$

where "+" between $(n+1)x$ and $f_n(x)$ denotes addition in $FA[-1,1]$. It is readily verified that ϕ is continuous and one-to-one. Thus if we put

$$\phi\left[-\frac{n+1}{n+2}, \frac{n+1}{n+2}\right] = Y_n \quad \text{then } \phi : \left[-\frac{n+1}{n+2}, \frac{n+1}{n+2}\right] \to Y_n$$

is a homeomorphism, for each n. Observing that (by Theorem A) $FA[-1,1]$ has k_ω-decomposition $FA[-1,1] = \bigcup gp_n[-1,1]$, that Y_n is compact, and that $gp_n[-1,1] \cap \phi(-1,1) = Y_{n-2}$, we see that $Y = \phi((-1,1))$ is a k_ω-space with k_ω-decomposition $Y = \bigcup Y_n$. It follows that ϕ is a closed map and hence is a homeomorphism of $(-1,1)$ onto Y. The proof is completed by the following lemma which, in particular, implies that $Y \setminus \{0\}$ is a free algebraic basis for the group it generates and that the conditions of Theorem B are satisfied. Hence $gp(Y)$ is $FA(Y)$, and $gp(Y)$ and Y are closed in $FA[-1,1]$, as required.

LEMMA 1. *With* $Y = \bigcup Y_n$ *as above, let* $w = \varepsilon_1 y_1 + \varepsilon_2 y_2 + \ldots + \varepsilon_n y_n$, *where, for each* i, $y_i \in Y \setminus \{0\}$ *and* $\varepsilon_i = \pm 1$, *and where* $y_i = y_j$ *implies* $i = j$. *Then* $w \notin gp_n[-1, 1]$. *If* $y_i \in Y_m \setminus Y_{m-1}$ *for some positive integer* m *and some* $i \in \{1, \ldots, n\}$ *then* $w \notin gp_m[-1, 1]$.

Proof. We shall use induction on n to prove that $w \notin gp_n[-1, 1]$. For $n = 1$ the result is trivially true, so suppose that the result holds for $n = p-1$, and consider a word $w = \varepsilon_1 y_1 + \ldots + \varepsilon_p y_p$, as in the statement of the lemma. For each i, write $y_i = \phi(x_i)$, for suitable elements $x_i \in (-1, 1)$. Assume without loss of generality that $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_p$.

If $x_i \in [-\frac{1}{2}, \frac{1}{2}]$ for each i, it is easy to check that $w = 2\varepsilon_1 x_1 + \ldots + 2\varepsilon_p x_p \notin gp_p[-1, 1]$. Otherwise, assume for convenience that $x_p > \frac{1}{2}$. (If $x_p \leqslant \frac{1}{2}$, we have $x_1 < -\frac{1}{2}$, and a similar argument applies.) We may then write

$$w = \varepsilon_1((k_1+1)x_1 + f_{k_1}(x_1)) + \ldots + \varepsilon_{p-1}((k_{p-1}+1)x_{p-1} + f_{k_{p-1}}(x_{p-1})) + \varepsilon_p((k_p+1)x_p + f_{k_p}(x_p))$$

$$= w' + \varepsilon_p((k_p+1)x_p + f_{k_p}(x_p)),$$

where $k_p \geqslant 1$ as $x_p > \frac{1}{2}$. It is clear that for each positive x_i, $f_{k_i}(x_i) \leqslant x_i$, while for each negative x_i, $f_{k_i}(x_i) < 0$. Therefore, since $x_i = x_p$ only if $\varepsilon_i = \varepsilon_p$, no occurrence of x_p is cancelled when we put w into its reduced form with respect to the free basis $[-1, 1] \setminus \{0\}$. Hence the reduced length of w is at least that of w', plus k_p. But by the inductive hypothesis, $w' \notin gp_{p-1}[-1, 1]$, so $w \notin gp_p[-1, 1]$, since $k_p \geqslant 1$. This completes the induction.

Now suppose that $y_i \in Y_m \setminus Y_{m-1}$ for some $i \in \{1, \ldots, n\}$. Again suppose without loss of generality that $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$. If $x_i \in [-\frac{1}{2}, \frac{1}{2}]$ we have $m = 0$, so the desired conclusion is clearly true. If $x_i \notin [-\frac{1}{2}, \frac{1}{2}]$, we may assume for convenience that $x_i > \frac{1}{2}$. Then $y_n \in Y_q \setminus Y_{q-1}$ for some $q \geqslant m$, and by the argument above, $(q+1)x_n$ appears in the reduced representation of w, and so $w \notin gp_q[-1, 1]$. In particular, $w \notin gp_m[-1, 1]$, and so the proof is complete.

It follows from Theorem B that if Y is a closed subspace of the k_ω-space X and if Y contains e, then the subgroup of $FA(X)$ generated by Y is $FA(Y)$.

Thus we obtain the

COROLLARY. *Let* Y *be any closed subspace of* $(0, 1)$. *Then* $FA(Y)$ *is topologically isomorphic to a closed subgroup of* $FA[0, 1]$.

EXAMPLES. (i) $FA[0, 1)$ is topologically isomorphic to a closed subgroup of $FA[0, 1]$.

(ii) Let Y be the subspace $\bigcup_{n=1}^{\infty} [2n, 2n+1]$ of $(0, \infty)$. Then $FA(Y)$ is topologically isomorphic to a closed subgroup of $FA[0, 1]$.

(iii) Let Z be the set of integers with the discrete topology. Then $FA(Z)$ is topologically isomorphic to a closed subgroup of $FA[0, 1]$.

§3. *The non-commutative case*

THEOREM 2 (Nickolas [4]). *$F(0, 1)$ is topologically isomorphic to a closed subgroup of $F[0, 1]$.*

Proof. The mapping ϕ of $(-1, 1)$ into $F[-1, 1]$ is defined in an analogous way to the mapping of $(-1, 1)$ into $FA[-1, 1]$ in Theorem 1. The proof of this theorem is exactly the same as the proof of Theorem 1, with the exception that the lemma requires a somewhat different argument.

LEMMA 2. *Let $Y = \bigcup Y_n$ be as in the proof of the above theorem. Let $w = y_1^{\varepsilon_1} y_2^{\varepsilon_2} \ldots y_n^{\varepsilon_n}$ where for each $i, y_i \in Y \setminus \{0\}$ and $\varepsilon_i = \pm 1$, and where $y_i = y_{i+1}$ implies $\varepsilon_i = \varepsilon_{i+1}$. Then $w \notin gp_n[-1, 1]$. If $y_i \in Y_m \setminus Y_{m-1}$ for some positive integer m and some $i \in \{1, \ldots, n\}$ then $w \notin gp_m[-1, 1]$.*

Proof. Writing $w = (x_1^{k_1+1} f_{k_1}(x_1))^{\varepsilon_1} \ldots (x_n^{k_n+1} f_{k_n}(x_n))^{\varepsilon_n}$, consider any pair of adjacent words

$$(x_i^{k_i+1} f_{k_i}(x_i))^{\varepsilon_i} (x_{i+1}^{k_{i+1}+1} f_{k_i}(x_{i+1}))^{\varepsilon_{i+1}}.$$

If $\varepsilon_i = \varepsilon_{i+1}$ there cannot be any cancellation. If $\varepsilon_i = 1$ and $\varepsilon_{i+1} = -1$ then we have

$$x_i^{k_i+1} f_{k_i}(x_i) f_{k_{i+1}}(x_{i+1})^{-1} x_{i+1}^{-k_{i+1}-1}.$$

Of course for these values of ε, $x_i \neq x_{i+1}$. So the only possible cancellation is of $f_{k_i}(x_i)$ with $f_{k_{i+1}}(x_{i+1})^{-1}$. Even if these do cancel, there is no more cancellation. The only other case is when $\varepsilon_i = -1$ and $\varepsilon_{i+1} = 1$. Then we have

$$f_{k_i}(x_i)^{-1} x_i^{-k_i-1} x_{i+1}^{k_{i+1}+1} f_{k_{i+1}}(x_{i+1})$$

and no cancellation occurs.

Thus the reduced form of w contains $x_i^{k_i+1}$ for $i = 1, \ldots, n$. The remainder of the proof of the lemma is routine.

COROLLARY. *Let Y be any closed subspace of $(0, 1)$. Then $F(Y)$ is topologically isomorphic to a closed subgroup of $F[0, 1]$.*

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