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ON SOMEWHAT DIVISIBLE GROUPS

BY

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Abstract. A group G is said to be somewhat divisible if it has divisible subgroups $H_1, \cdots, H_n$ such that $G = H_1 \cdot H_2 \cdots H_n$. It is observed that every connected locally compact Hausdorff topological group is somewhat divisible but not necessarily divisible. However, it is shown that a compact Hausdorff group is divisible if and only if it is somewhat divisible.

Introduction. A standard result ([5, 3, p. 101]) says that a compact Hausdorff topological group is connected if and only if it is divisible. In this note we introduce the weaker property "somewhat divisible", and show that for compact Hausdorff groups it is equivalent to "divisible". It is also shown that every connected locally compact Hausdorff group is somewhat divisible. For abelian groups the conditions "divisible" and "somewhat divisible" are equivalent and so this statement reduces to the familiar one which says that connected locally compact Hausdorff abelian groups are divisible. However, examples are known (see, for example, [1, p. 394]) of connected locally compact Hausdorff groups which are not divisible.

In [4] it was proved that any homomorphism ϕ from a connected locally compact Hausdorff group G into a free product $H = \Pi^*_{i \in I} H_i$ of groups must satisfy $\phi(G) \subseteq h^{-1} H_i h$, for some $i \in I$ and $h \in H$. We can now deduce this from the purely algebraic result that any homomorphism from a somewhat divisible group G into a free product maps G into a conjugate of one of the factors.

Results. Definition. A group G is said to be *somewhat divisible* if it has divisible subgroups $H_1, \cdots, H_n$ such that $G = H_1 \cdot H_2 \cdots H_n$; that is, each g in G satisfies $g = h_1 h_2 \cdots h_n$, for some $h_i \in H_i$, $i = 1, \cdots, n$.

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Clearly any divisible group is somewhat divisible. However, the following theorem will yield that somewhat divisible groups are not necessarily divisible.

PROPOSITION 1. *Every connected locally compact Hausdorff group is somewhat divisible.*

Proof. By the Iwasawa Structure Theorem [3, p. 118] any connected locally compact Hausdorff group G has subgroups $R_1, \cdots, R_n$ and K such that $G = R_1 \cdots R_n K$, each R_i is topologically isomorphic to the additive group of real numbers with the usual topology, and K is a compact connected group. As every compact connected Hausdorff group is divisible [5], K is divisible. As each R_i is also divisible, G is expressed as a product of divisible subgroups and so is somewhat divisible.

As remarked in the Introduction there are examples of connected locally compact groups which are not divisible and so we see that the class of somewhat divisible groups properly contains the class of divisible groups.

PROPOSITION 2. *A compact Hausdorff group G is divisible if and only if it is somewhat divisible.*

Proof. Obviously if G is divisible, then it is somewhat divisible.

Assume G is somewhat divisible. Then it has subgroups $H_1, \cdots, H_n$ which are divisible and such that $G = H_1 \cdot H_2 \cdots H_n$. Let $\bar{H}_i$ be the closure in G of H_i, $i \in \{1, \cdots, n\}$. Let $h \in \bar{H}_i$ and m be any positive integer. Then there is a net h_α of elements of H_i which converges to h. As H_i is divisible for each i, there is an element x_α in H_i such that $(x_\alpha)^m = h_\alpha$, for each α. As x_α is a net in the compact group $\bar{H}_i$, it must have a convergent subnet x_δ. So x_δ converges to an element $x \in \bar{H}_i$. Clearly $(x_\delta)^m$ converges to x^m. However, $(x_\delta)^m = h_\delta$ is a subnet of h_α and so converges to h. As G is Hausdorff, the net h_α cannot converge to two distinct elements x^m and h, so we must have $x^m = h$. So every element of $\bar{H}_i$ has at least one m^{th} root. Hence $\bar{H}_i$ is divisible. So $\bar{H}_i$ is a divisible compact group and is therefore [5] connected. Now $G = H_1 \cdot H_2 \cdots H_n$ implies $G = \bar{H}_1 \cdot \bar{H}_2 \cdots \bar{H}_n$. As each $\bar{H}_i$ is connected,

G must be connected. Thus G is a connected compact Hausdorff group, and consequently is divisible.

The following three Propositions have obvious proofs.

PROPOSITION 3. *An abelian group is divisible if and only if it is somewhat divisible.*

PROPOSITION 4. *If $G_1,\cdots,G_n$ are somewhat divisible groups, then $G_1 \times G_2 \times \cdots \times G_n$ is somewhat divisible.*

PROPOSITION 5. *Any quotient group of a somewhat divisible group is somewhat divisible.*

It is an easy consequence of the Kurosh Subgroup Theorem [2] that a divisible subgroup G of a free product $H = \Pi^*_{i\in I} H_i$ of groups H_i must satisfy $G \subseteq h^{-1}H_i h$, for some $h \in H$ and $i \in I$. We now prove the analogous result for somewhat divisible groups.

PROPOSITION 6. *Let G be a subgroup of a free product $H = \Pi^*_{i\in I} H_i$ of groups H_i. If G is somewhat divisible, then there exists an $h \in H$ and an $i \in I$ such that $G \subseteq h^{-1}H_i h$.*

Proof. As G is somewhat divisible, it has divisible subgroups $D_1, D_2,\cdots, D_n$ such that $G = D_1 \cdot D_2 \cdots D_n$. As each D_j is divisible, there exists an $i \in I$ such that $D_j \subseteq h_i^{-1} H_i h_i$ for some $h_i \in H$.

Now each element $h \in H$ can be written in the reduced form $h = h_{j_1} h_{j_2} \cdots h_{j_k}$, where $h_{j_l} \in H_l$. We refer to the number k as the length of the element h with respect to $\{H_i : i \in I\}$. From the remark in the above paragraph we see that the set of elements in each D_j has bounded length with respect to $\{H_i : i \in I\}$. From this it is immediately seen that the set of elements in G has bounded length with respect to $\{H_i : i \in I\}$. However, it is known [2], that any subgroup of a free product having its elements of bounded length must lie in a conjugate of one of the factors of the free product. So we have the required result.

We can now deduce Theorem 2 of [4].

COROLLARY. *Let G be a connected locally compact Hausdorff group and ϕ a homomorphism of G into a free product $H = \Pi^*_{i\in I} H_i$ of groups H_i. Then there exists an $i \in I$ and $h \in H$ such that $G \subseteq h^{-1} H_i h$.*

Proof. By Proposition 1, G is somewhat divisible. Proposition 5 then says that $\phi(G)$ is somewhat divisible. As $\phi(G)$ is a subgroup of $\Pi^*_{i\in I} H_i$, the required result then follows from Proposition 6.

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