NORMS ON FREE TOPOLOGICAL GROUPS

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ABSTRACT

A norm on a group G is a function N mapping G into the set of non-negative real numbers such that for each x and y in G, $N(xy^{-1}) \leqslant N(x)+N(y)$ and $N(e) = 0$, where e is the identity element of G. It is shown here that if $F(X)$ is the free topological group on any completely regular Hausdorff space X and H is a subgroup of $F(X)$ generated by a finite subset of X, then any norm on H can be extended to a continuous norm on $F(X)$.

Introduction

In [1] Abels gave a difficult proof of the fact that if $F(X)$ is the (Markov) free topological group on a completely regular Hausdorff space X and H is a subgroup of $F(X)$ generated by a finite subset of X, then there is a continuous norm N on $F(X)$ such that for each $h \in H$, $N(h)$ is approximately equal to the reduced length of h with respect to X. Our work shows that actual equality can be achieved.

Preliminaries. A *norm* on a group G is a function N mapping G into the set of non-negative real numbers such that $N(xy^{-1}) \leqslant N(x)+N(y)$ for each x and y in G, and $N(e) = 0$ where e is the identity element of G. We note that if N is a norm on a group G and ϕ is a homomorphism of a group H into G, then $N\phi$ is a norm on H. Also if $\{N_i : i \in I\}$ is a family of norms on a group G such that $N(g) = \sup_{i \in I}(N_i(g))$ exists for each $g \in G$, then N is a norm on G.

Our main tool will be a construction due to Hartman and Mycielski [4] which embeds any topological group in a path-connected topological group. Given a topological group G, the set G^* is defined to consist of those functions f from the half-open interval $[0, 1)$ into G such that for some set $\{a_0, a_1, \ldots, a_{n+1}\}$ with

$$0 = a_0 < a_1 < a_2 < \ldots < a_n < a_{n+1} = 1,$$

f is constant on each interval $[a_i, a_{i+1})$. Then G^* becomes a group if the operations are defined on it pointwise. Further, G^* becomes a topological group if the sub-basic open neighbourhoods of each $f \in G^*$ are defined to be

$$\{h : h \in G^* \text{ and } |\{x : h(x) \notin Vf(x)\}| < \varepsilon\},$$

where V is any open neighbourhood of e in G, $|.|$ is the Lebesgue measure on $[0, 1)$ and $\varepsilon > 0$. Hartman and Mycielski observed that G^* is path-connected and that G is topologically isomorphic to the closed subgroup of G^* consisting of the constant functions from $[0, 1)$ to G. Brown and Morris [2] indicated how this construction can be extended from topological groups to topological spaces and showed the following result (which is a consequence of their Propositions 6 and 7).

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Proposition. Let G be any topological group and G^* be as above. If ρ is any bounded continuous pseudo-metric on G, then d, given by

$$d(f, g) = \int_0^1 \rho(f(t), g(t))dt$$

for any f and g in G^*, is a continuous pseudo-metric on G^* which agrees with ρ on G. If ρ is a metric then d is a metric. If ρ is left-invariant then d is left-invariant.

Remark. With regard to the condition "bounded" in the above proposition, there is a little confusion in the literature. Hartman and Mycielski [4] claimed, without proof, that if G is a metric topological group with metric ρ then G^* is a metric topological group with metric d given by

$$d(f, g) = \int_0^1 \rho(f(t), g(t))dt.$$

In the case that ρ is bounded Brown and Morris [2] proved that the topology induced by the metric d and the previously defined topology on G^* coincide. The example below shows that this is not the case if ρ is unbounded.

Let (X, ρ) be any metric space such that ρ is unbounded. Let $x \in X$ and x_n, $n = 1, 2, \ldots$, a sequence of elements of X such that $\rho(x, x_1) > 1$ and

$$\rho(x, x_n) + 1 < \rho(x, x_{n+1}).$$

For each positive integer n, we define $f_n \in X^*$ by

$$f_n(t) = \begin{cases} x_n, & 0 \leqslant t < \dfrac{1}{\rho(x, x_n)} \\ x, & \dfrac{1}{\rho(x, x_n)} \leqslant t < 1. \end{cases}$$

Let d be the metric on X^* given by

$$d(f, g) = \int_0^1 \rho(f(t), g(t))dt,$$

for f and g in X^*. We claim that d is not a continuous metric on X^*. In particular, we show that the set $\{f_n; n = 1, 2, \ldots\}$ is closed with respect to d but not closed in X^*.

To describe the topology of X^* we introduce the bounded metric d_1 on X^* by putting

$$d_1(f, g) = \int_0^1 \left[\min\Big(\rho(f(t), g(t)), 1\Big)\right]dt$$

for f and g in X^*. By the work of Brown and Morris [2], the metric d_1 induces the correct topology on X^*.

Clearly

$$d_1(f_n, f_x) = \frac{1}{\rho(x, x_n)} < \frac{1}{n}$$

if f_x is the constant function mapping [0, 1) into $x \in X$. So $\{f_n : n = 1, 2, \ldots\}$ is not closed in X^*. However if $f \in X^* \setminus \{f_n ; n = 1, 2, \ldots\}$ then either $f = f_x$ or for some a and b in [0, 1) with $a \neq b, f[a, b) = x' \neq x$. If $f = f_x$ then $d(f_n, f_x) = 1$. If $f \neq f_x$, choose an m such that

$$\frac{1}{\rho(x, x_m)} < \frac{b}{2};$$

in which case, for all $n > m$, $d(f_n, f) \geqslant (b/2)\, \rho(x, x')$. Hence $\{f_n; n = 1, 2, \ldots\}$ is closed with respect to d but not closed in X^*.

Results. THEOREM. *Let Y be a completely regular Hausdorff space and $F(Y)$ the (Markov) free topological group on Y. If H is a subgroup of $F(Y)$ generated by a subset $X = \{x_1, \ldots, x_n\}$ of Y, then any norm N on H extends to a continuous norm $\bar{N}$ on $F(Y)$.*

Proof. We show that the result is true in the case $Y = [0, 1]$ and then show that this implies the result as stated.

Let $X = \{x_1, \ldots, x_n\}$ be a subset of [0, 1] with each $x_i < x_{i+1}$ and N a norm on the subgroup H of $F[0, 1]$ generated by X. The pseudo-metric ρ defined on H by $\rho(u, v) = N(v^{-1}u)$ for u and v in H, is left-invariant. For each positive integer k, denote by $gp_k(X)$ the subset of H consisting of all words of reduced length not greater than k with respect to X. Put $m_k = \max\{N(x) : x \in gp_k(X)\}$ and $\rho_k(u, v) = \min\{m_k, \rho(u, v)\}$ for each u and v in H. Then $\{\rho_1, \rho_2, \ldots, \rho_n, \ldots\}$ is a family of bounded left-invariant pseudo-metrics on H. By [6, Theorem 1.11] any finitely generated subgroup of a free topological group is discrete and so each ρ_k is continuous. If H^* is defined in the manner described in the Preliminaries, then by the Proposition there, each ρ_k can be extended to a left-variant pseudo-metric d_k which is continuous on H^* and is given by

$$d_k(f, g) = \int_0^1 \rho_k(f(t), g(t))dt,$$

for each f and g in H^*. For each k, define a continuous norm N_k on H^* by putting $N_k(f) = d_k(f, e)$ where $f \in H^*$ and e is the identity of H^*.

We define a continuous map $\phi : [0, 1] \to H^*$ as follows: if $1 \geqslant x \geqslant x_n$ then $\phi(x)(t) = x_n$, for every $t \in [0, 1)$; if $0 \leqslant x \leqslant x_1$, then $\phi(x)(t) = x_1$, for every $t \in [0, 1)$; if $x = rx_i + (1-r)x_{i+1}$, for $0 \leqslant r \leqslant 1$ and some $i \in \{1, 2, \ldots, n\}$, then $\phi(x)(t) = x_i$, for $0 \leqslant t < r$ and $\phi(x)(t) = x_{i+1}$ for $r \leqslant t < 1$. Now, by the freeness of $F[0, 1]$ there exists a continuous homomorphism $\Phi : F[0, 1] \to H^*$ which extends ϕ. Then $\{N_1\Phi, N_2\Phi, \ldots\}$ is a sequence of continuous norms on $F[0, 1]$. We define $\bar{N}$ on $F[0, 1]$ by putting $\bar{N}(w) = \sup\{N_1\Phi(w), N_2\Phi(w), \ldots\}$ for each $w \in F[0, 1]$, and claim that $\bar{N}$ is the required norm.

Firstly observe that for each $x \in [0, 1]$ and $t \in [0, 1)$, $\phi(x)(t) \in X$. So if the word w in $F[0, 1]$ has reduced length ℓ with respect to the free basis [0, 1], then $\Phi(w)(t) \in gp_\ell(X)$, for each $t \in [0, 1]$. So

$$N_k\Phi(w) = d_k(\Phi(w), e) = d_\ell(\Phi(w), e) \text{ if } k \geqslant \ell$$

and thus $N_k\Phi(w) \leqslant m_\ell$, for $k = 1, 2, \ldots$. Hence the supremum exists and $\overline{N}$ is indeed a norm on $F[0, 1]$ which extends the norm N on H. Further the map $\overline{N}$ restricted to $gp_k[0, 1]$, that is $\overline{N}|gp_k[0, 1]$, equals $N_k\Phi|gp_k[0, 1]$ and so $\overline{N}|gp_k[0, 1]$ is continuous, for each k. As $F[0, 1]$ has the weak topology with respect to $\{gp_k[0, 1] : k = 1, 2, \ldots\}$. see for example [3; Theorem 4], $\overline{N}$ is continuous. Thus the result is true for $Y = [0, 1]$.

Now let Y be any completely regular Hausdorff space and H a subgroup of $F(Y)$ generated by a finite subset $X = \{x_1, \ldots, x_n\}$ of Y. Further, let N be any norm on H. We define $X' = \{x_1', \ldots, x_n'\}$ to be any subset of $[0, 1]$ having n elements and H' the subgroup of $F[0, 1]$ generated by X'. By [5, Theorem 3.6] there exists a continuous map $\theta : Y \to [0, 1]$ such that $\theta(x_i) = x_i'$ for each i. So there exists a continuous homomorphism $\delta : F(Y) \to F[0, 1]$ such that δ extends θ. Now we can define a norm N' on H' by putting $N'(h') = N(h)$, where $h' \in H'$ and h is the unique element of H such that $\theta(h) = h'$. By the argument above, N' can be extended to $\overline{N}'$, a norm continuous on $F[0, 1]$. We now define $\overline{N}$ on $F(Y)$ to be $\overline{N}'\delta$. Clearly $\overline{N}$ is a continuous norm on $F(Y)$ and it agrees with N on H, which completes the proof.

Remark. We note that the norm $\overline{N}$ constructed in the proof of the Theorem has the property that

$$\sup\{\overline{N}(w) : w \in gp_k(Y)\} = \sup\{N(x) : x \in gp_k(X)\}$$

In the particular case that the norm is the length norm we obtain:

COROLLARY 1. *Let Y be a completely regular Hausdorff space and $F(Y)$ the free topological group on Y. If H is a subgroup of $F(Y)$ generated by a finite subset of Y, then there is a continuous norm N on $F(Y)$ such that for each $h \in H$, $N(h)$ equals the reduced length of h with respect to Y. Further, $N(y) \leqslant 1$ for all y in Y.*

Definition. Let N be a norm on a group G. Then N is said to be a *proper norm* if $N(g)$ equals zero if and only if g is the identity element of G.

COROLLARY 2. *The norm N described in Corollary 1 can be chosen to be proper if and only if Y admits a continuous metric.*

Proof. If Y admits a continuous metric ρ, then without loss of generality ρ can be assumed to be bounded by 1. From the work of Graev [3, Theorem 1] ρ can be extended to a two-sided invariant metric ρ' on $F(Y)$. So ρ' induces a continuous proper norm N' on $F(Y)$. The required norm is then the maximum of N' and the one arising from the proof of the Theorem.

Conversely if $F(Y)$ admits a proper continuous norm, then it admits a continuous metric and hence the subspace Y does also.

The result stated in our third corollary is not new, but it is of some interest that it is a consequence of our Theorem.

COROLLARY 3. *Let Y be any completely regular Hausdorff space and $F(Y)$ the free topological group on Y. Then $gp_n(Y)$ is closed in $F(Y)$ for each n.*

Proof. For each pair of points x_i, x_j in Y there is a continuous norm N_{ij} on $F(Y)$ which is the extension of the length norm on the subgroup generated by $\{x_i, x_j\}$. We readily see that

$$gp_n(Y) = \cap N_{ij}^{-1}[0, n]$$

which is closed.

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