

Locally compact products and coproducts in categories of topological groups

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In the category of locally compact groups not all families of groups have a product. Precisely which families do have a product and a description of the product is a corollary of the main theorem proved here. In the category of locally compact abelian groups a family $\{G_j; j \in J\}$ has a product if and only if all but a finite number of the G_j are of the form $K_j \times D_j$, where K_j is a compact group and D_j is a discrete torsion free group. Dualizing identifies the families having coproducts in the category of locally compact abelian groups and so answers a question of Z. Semadeni.

It is well known that if C is the category of all topological groups (hereinafter referred to as $TopGps$), of all abelian topological groups, of all compact groups, or of all compact abelian groups with the morphisms being the continuous homomorphisms then, for any family $\{G_j; j \in J\}$ of

members of C , both the product $\prod_{j \in J}^C G_j$ and the coproduct $\coprod_{j \in J}^C G_j$

exist. However Semadeni [2, 3] notes that very little is known about coproducts (he could also have said products) in the category LCA of

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locally compact abelian groups. It is clear that if $\{G_j; j \in J\}$ is a family in LCA such that all but a finite number of G_j are discrete,

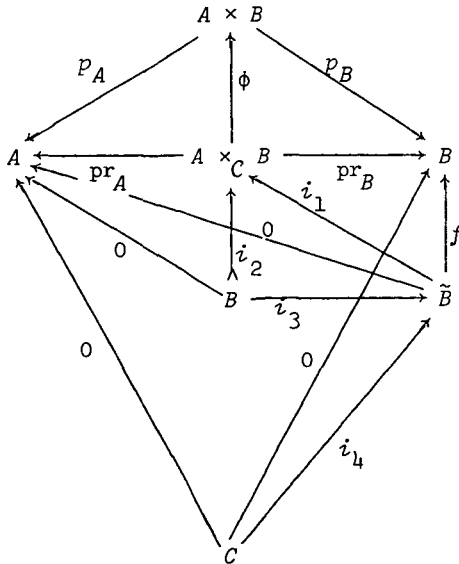
then $\prod_{j \in J}^{LCA} G_j$ exists. Semadeni [3, Problem 490] asks if there is an infinite family $\{G_j; j \in J\}$ of non-discrete members of LCA such that

$\prod_{j \in J}^{LCA} G_j$ exists. We give an affirmative answer and characterize those

families $\{G_j; j \in J\}$ for which $\prod_{j \in J}^{LCA} G_j$ exists. This is done by solving the dual problem of describing products in LCA which is a corollary of our main result.

PROPOSITION 1. *If C is a full subcategory of $TopGps$ which has finite limits, then the finite products in C are the finite products in $TopGps$.*

Proof. It suffices to prove that if A and B are in C , then their product in C , $A \times_C B$, is topologically isomorphic to $A \times B$, their product in $TopGps$.



Let pr_A and pr_B be the projections of $A \times_C B$ into A and B ,

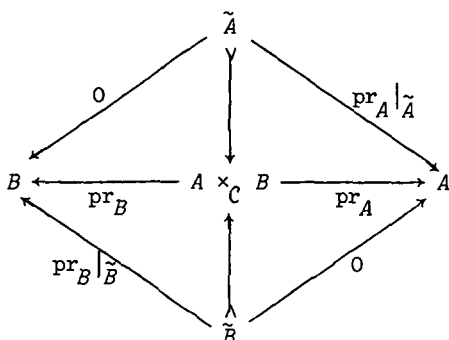
respectively. Also let p_A and p_B be the projections of $A \times B$ into A and B respectively. As $A \times B$ is the product in $TopGps$, there exists a continuous homomorphism $\phi : A \times_C B \rightarrow A \times B$ such that $pr_A = p_A \phi$ and $pr_B = p_B \phi$. Let \tilde{B} be the kernel of pr_A , and i_1 the inclusion of \tilde{B} in $A \times_C B$. Let $f : \tilde{B} \rightarrow B$ be $pr_B i_1$.

The identity map $1 : B \rightarrow B$ and the trivial map $0 : B \rightarrow A$ imply the existence of a continuous homomorphism i_2 of B into $A \times_C B$ which is a topological embedding. As $pr_A i_2 = 0$, the definition of \tilde{B} implies the existence of a continuous homomorphism $i_3 : B \rightarrow \tilde{B}$ which is a topological embedding.

Let C be the kernel of f . So i_4 is the inclusion map of C in \tilde{B} and $f i_4$ is the trivial map of C into B . Mapping C into \tilde{B} by i_4 and \tilde{B} into A by 0 yields the trivial map $0 : C \rightarrow A$. The pair of maps $0 : C \rightarrow A$ and $0 : C \rightarrow B$ yield a unique continuous homomorphism $\theta : C \rightarrow A \times_C B$ such that $pr_A \theta = 0$ and $pr_B \theta = 0$. But there are two maps with this property, $0 : C \rightarrow A \times_C B$ and $i_1 i_4 : C \rightarrow A \times_C B$. So $i_4 = 0$ and thus C is the trivial group. Hence f is injective. But as $f i_3 = 1$, f is also a retraction and hence f is a topological group isomorphism and \tilde{B} is topologically isomorphic to B .

Similarly, if we put \tilde{A} equal to the kernel of pr_B , then we can show $pr_A|_{\tilde{A}}$ is a topological isomorphism of \tilde{A} onto A .

We show next that $A \times_C B = \tilde{A}\tilde{B}$ and $\tilde{A} \cap \tilde{B} = \{1\}$;



Let $g \in \tilde{A} \cap \tilde{B}$. As $g \in \tilde{B}$, $\text{pr}_A(g) = 1$. But $\text{pr}_A|_{\tilde{A}}$ is an isomorphism and so $g = 1$. So $\tilde{A} \cap \tilde{B} = \{1\}$.

Now if $x \in A \times_C B$, then $h = (\text{pr}_A|_{\tilde{A}})^{-1} \text{pr}_A(x) \in \tilde{A}$. So $\text{pr}_A(h) = \text{pr}_A(x)$. Thus $\text{pr}_A(h^{-1}x) = 1$, which implies $h^{-1}x \in \ker(\text{pr}_A) = \tilde{B}$. Hence $x \in \tilde{A}\tilde{B}$ and $A \times_C B = \tilde{A}\tilde{B}$.

As \tilde{A} and \tilde{B} are normal subgroups of $A \times_C B$ and $A \times_C B = \tilde{A}\tilde{B}$ and $\tilde{A} \cap \tilde{B} = \{1\}$, we have that $A \times_C B$ is algebraically isomorphic to $A \times B$. More precisely ϕ is an algebraic isomorphism of $A \times_C B$ onto $A \times B$.

Now define $\psi : A \times B \rightarrow A \times_C B$ by

$$\psi(a, b) = \left[(\text{pr}_A|_{\tilde{A}})^{-1} a \right] \left[(\text{pr}_B|_{\tilde{B}})^{-1} b \right].$$

If $g \in A \times_C B$, then $g \in \tilde{A}\tilde{B}$; that is, $g = \tilde{a}\tilde{b}$ with $\tilde{a} \in \tilde{A}$ and $\tilde{b} \in \tilde{B}$. Then $\phi(g) = (\text{pr}_A(\tilde{a}), \text{pr}_B(\tilde{b}))$. Clearly ϕ and ψ are inverse maps. As ψ is also continuous, ϕ is a topological group isomorphism and $A \times_C B$ is topologically isomorphic to $A \times B$.

REMARKS. It is not necessary to assume \mathcal{C} is a full subcategory of TopGrps in Proposition 1. It suffices to assume that \mathcal{C} has identity arrows, 0 arrows, and inclusion arrows.

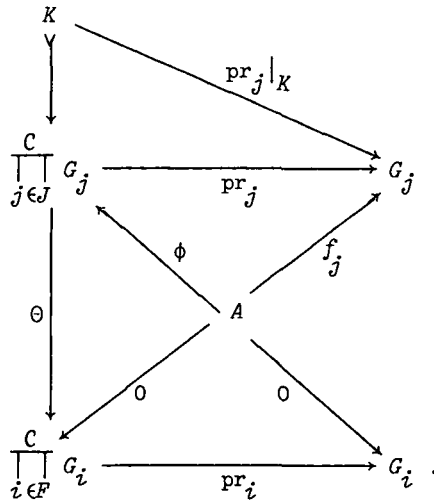
The reader should be cautioned not to assume *a priori* that, in proofs like the preceding and in the discussions which follow, the products we consider are defined on set theoretical cartesian products.

LEMMA 1. Let C be a full subcategory of $TopGps$ which has finite limits. If F is a finite subset of a set J and $\prod_{j \in J}^C G_j$ exists, then

$\prod_{j \in J \setminus F}^C G_j$ exists and

$$\prod_{j \in J}^C G_j \simeq \prod_{j \in F}^C G_j \times \prod_{j \in J \setminus F}^C G_j .$$

Proof. Let K be the kernel of the canonical map θ of $\prod_{j \in J}^C G_j$ into $\prod_{j \in F}^C G_j$. We claim that $K = \prod_{j \in J \setminus F}^C G_j$, with the projections of K into G_j being the projections of $\prod_{j \in J}^C G_j$ (for $j \in J \setminus F$) restricted to K .



Let A be any member of C and f_j , $j \in J \setminus F$, a family of continuous homomorphisms of A into G_j . Define $f_i : A \rightarrow G_i$, $i \in F$, to be the trivial homomorphisms. Then there exists a continuous homomorphism ϕ of A into $\prod_{j \in J}^C G_j$ such that $pr_j \phi = f_j$, $j \in J$. As

$\text{pr}_i \phi = 0$, $i \in F$, $\Theta \phi = 0$, and so $\phi(A) \subseteq K$. Hence $K = \prod_{j \in J \setminus F}^C G_j$.

Once we know $\prod_{j \in J \setminus F}^C G_j$ exists, the usual categorical argument shows that

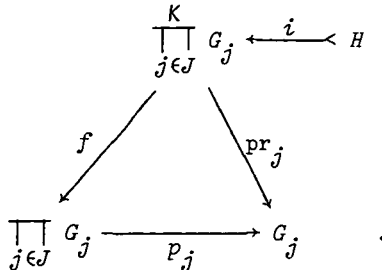
$\prod_{j \in J}^C G_j$ is topologically isomorphic to $\prod_{j \in F}^C G_j \times_C \prod_{j \in J \setminus F}^C G_j$, which by

Proposition 1, is the required result.

PROPOSITION 2. *Let \mathcal{K} be a full subcategory of TopGps which is complete and whose objects are compact. If every compact subgroup of a topological group in \mathcal{K} is also in \mathcal{K} , then for any family $\{G_j; j \in J\}$*

in \mathcal{K} , $\prod_{j \in J}^{\mathcal{K}} G_j \simeq \prod_{j \in J} G_j$; that is, the product in \mathcal{K} and the product in TopGps coincide.

Proof.



Let p_j and pr_j be the projections of $\prod_{j \in J} G_j$ into G_j and

$\prod_{j \in J}^{\mathcal{K}} G_j$ into G_j , respectively. As $\prod_{j \in J} G_j$ is the product in TopGps of

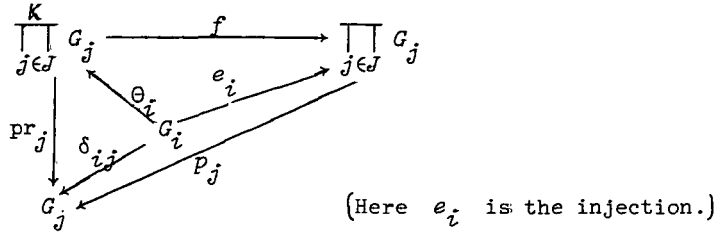
$\{G_j; j \in J\}$, the maps $\text{pr}_j : \prod_{j \in J}^{\mathcal{K}} G_j \rightarrow G_j$ induce a continuous homomorphism

$f : \prod_{j \in J}^{\mathcal{K}} G_j$ into $\prod_{j \in J} G_j$ such that $p_j f = \text{pr}_j$, for each $j \in J$.

Let $x \in \prod_{j \in J}^{\mathcal{K}} G_j$ be such that $f(x) = 1$. Then $p_j f(x) = 1$ and so $\text{pr}_j(x) = 1$, for each $j \in J$. Let H be the closed subgroup of $\prod_{j \in J}^{\mathcal{K}} G_j$,

generated by $\{x\}$. Then $\text{pr}_j(H) = \{1\}$, for all $j \in J$. If

$i : H \rightarrow \prod_{j \in J}^K G_j$ is the inclusion map, then $\text{pr}_j i = 0 = \text{pr}_j 0'$, where 0 is the trivial map of H into G_j and $0'$ is the trivial map of H into $\prod_{j \in J}^K G_j$. As $\prod_{j \in J}^K G_j$ is the product in K , the uniqueness in the product property implies that $i = 0'$. Thus $H = \{1\}$ and $x = 1$. Hence the map f is injective.



Now let $\delta_{ij} : G_i \rightarrow G_j$ be the identity map if $i = j$ and the trivial homomorphism if $i \neq j$. Then there exist continuous homomorphisms

$$\Theta_i : G_i \rightarrow \prod_{j \in J}^K G_j \text{ such that } \delta_{ij} = \text{pr}_j \Theta_i \text{ and } \delta_{ij} = p_j e_i .$$

$j \in J$. By uniqueness in the product property $f \Theta_i = e_i$. So $f \left(\prod_{j \in J}^K G_j \right)$ is a compact subgroup of $\prod_{j \in J} G_j$ which contains $e(G_i)$ for each $i \in J$.

Thus $f \left(\prod_{j \in J}^K G_j \right) = \prod_{j \in J} G_j$ and f is surjective. Hence f is a

topological group isomorphism of $\prod_{j \in J}^K G_j$ onto $\prod_{j \in J} G_j$.

LEMMA 2. Let C be a full subcategory of TopGps and K a full subcategory of C which is complete and whose objects are all compact subgroups of topological groups in C . If $K_j \in K$, for $j \in J$, then

$$\prod_{j \in J}^C K_j \text{ exists and is topologically isomorphic to } \prod_{j \in J}^K K_j .$$

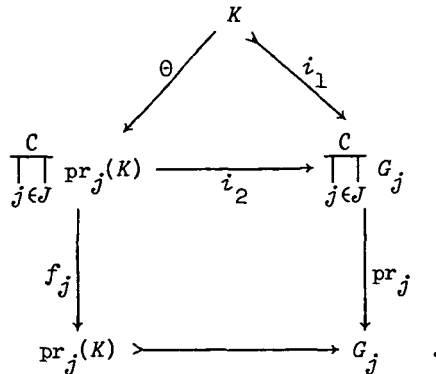
Proof. This follows immediately from the fact that $\prod_{j \in J}^K K_j$ is $\prod_{j \in J} K_j$, the product in *TopGps* of $\{K_j : j \in J\}$.

LEMMA 3. Under the hypotheses of Lemma 2, if $K_j \leq G_j$, where $K_j \in K$ and $G_j \in C$ for $j \in J$ and $\prod_{j \in J}^C G_j$ exists, then the canonical homomorphism of $\prod_{j \in J}^C K_j$ into $\prod_{j \in J}^C G_j$ is a topological embedding.

Proof. This follows immediately from the fact that $\prod_{j \in J}^C K_j$ is $\prod_{j \in J} K_j$, and so is compact.

LEMMA 4. Let C be a full subcategory of *TopGps* with finite limits and K a full subcategory of compact C -objects which is complete and contains all compact subgroups of groups in C . If $G_j \in C$, $j \in J$, and $\prod_{j \in J}^C G_j$ exists and K is a compact subgroup of $\prod_{j \in J}^C G_j$, then $K \leq \prod_{j \in J}^C \text{pr}_j(K) \leq \prod_{j \in J}^C G_j$, where pr_j is the canonical projection of $\prod_{j \in J}^C G_j$ into G_j , $j \in J$.

Proof.



As K is compact, $\text{pr}_j(K)$ is compact and so is $\prod_{j \in J} \text{pr}_j(K)$. Hence $\prod_{j \in J} \text{pr}_j(K)$ exists and by Lemma 2 is isomorphic to $\prod_{j \in J}^C \text{pr}_j(K)$. By Lemma 3, the canonical homomorphism $\prod_{j \in J}^C \text{pr}_j(K) \rightarrow \prod_{j \in J}^C G_j$ is a topological embedding i_2 . The maps $\text{pr}_j : K \rightarrow \text{pr}_j(K)$ imply the existence of a continuous homomorphism $\theta : K \rightarrow \prod_{j \in J}^C \text{pr}_j(K)$ such that $f_j \theta = \text{pr}_j$, where f_j are the projections of $\prod_{j \in J}^C \text{pr}_j(K)$. Finally observe that $i_2 \theta = i_1$, the inclusion of K in $\prod_{j \in J}^C G_j$. So θ is a topological embedding, as required.

LEMMA 5. *If H_j and G_j , $j \in J$, are topological groups with $H_j \leq G_j$ and $\prod_{j \in J} H_j$ an open subgroup of $\prod_{j \in J} G_j$, then there is a finite subset F of J such that for all $j \in J \setminus F$, $H_j = G_j$.*

Proof. This follows from the definition of the Tychonoff product topology.

LEMMA 6. *Let C be a full subcategory of TopGps which has finite limits. If $G_j \in C$, $j \in J$, and $\prod_{j \in J}^C G_j$ exists and is locally compact, then each G_j is locally compact and there exists a finite subset F of J such that for $j \in J \setminus F$, G_j has the component of the identity compact.*

Proof. By Proposition 1 and Lemma 1, each G_j is locally compact. If none of G_{j_1}, \dots, G_{j_n} has compact component then each contains a copy of R , the additive group of real numbers with the usual topology.

Proposition 1 and Lemma 1 then imply that $G_{j_1} \times G_{j_2} \times \dots \times G_{j_n} \leq \prod_{j \in J}^C G_j$.

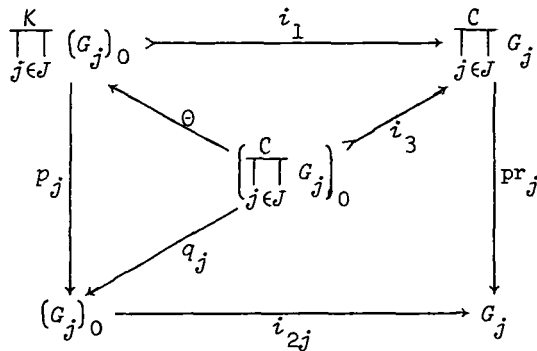
So $\prod_{j \in J}^C G_j$ has a subgroup topologically isomorphic to R^n . But no

locally compact group can contain R^n , for arbitrary large n . So all but a finite number of the G_j have compact components.

LEMMA 7. *Let C be a full subcategory of TopGps with finite limits and K a full subcategory of compact C -objects which is complete and contains all compact subgroups of groups in C , and let $G_j \in C$, $j \in J$ be such that each G_j has its connected component $(G_j)_0$ compact.*

If $\prod_{j \in J}^C G_j$ exists then its component $\left(\prod_{j \in J}^K (G_j)_0\right)_0$ is topologically isomorphic to $\prod_{j \in J}^K (G_j)_0$.

Proof. By Lemma 3, $\prod_{j \in J}^K (G_j)_0 \leq \prod_{j \in J}^C G_j$, and as $\prod_{j \in J}^K (G_j)_0$ is connected, $\prod_{j \in J}^K (G_j)_0 \leq \left(\prod_{j \in J}^C G_j\right)_0$.



Let i_1 be the inclusion map of $\prod_{j \in J}^K (G_j)_0$ in $\prod_{j \in J}^C G_j$, i_2j the inclusion map of $(G_j)_0$ in G_j , pr_j the projection of $\prod_{j \in J}^C G_j$ into G_j , i_3 the inclusion map of $\left(\prod_{j \in J}^C G_j\right)_0$ in $\prod_{j \in J}^C G_j$, q_j the

restriction of pr_j to $\left(\prod_{j \in J}^C G_j\right)_0$ and p_j the projection of $\prod_{j \in J}^K (G_j)_0$ into $(G_j)_0$. As $\left(\prod_{j \in J}^C G_j\right)_0$ is connected and q_j is continuous, $q_j \left[\left(\prod_{j \in J}^C G_j\right)_0 \right] \subseteq (G_j)_0$. The family of maps q_j determine a continuous homomorphism $\theta : \left(\prod_{j \in J}^C G_j\right)_0 \rightarrow \prod_{j \in J}^K (G_j)_0$ such that $p_j \theta = q_j$ for each $j \in J$. The family of maps $i_{2j} q_j$ of $\left(\prod_{j \in J}^C G_j\right)_0$ into G_j , $j \in J$, determine a unique continuous homomorphism $\delta : \left(\prod_{j \in J}^C G_j\right)_0 \rightarrow \prod_{j \in J}^C G_j$ such that $\text{pr}_j \delta = i_{2j} q_j$. But the maps i_3 and $i_1 \theta$ both have this property and thus $i_3 = i_1 \theta$. Hence $\left(\prod_{j \in J}^C G_j\right)_0 = \prod_{j \in J}^K (G_j)_0$.

LEMMA 8. *Under the hypotheses of Lemma 7, if $\prod_{j \in J}^C G_j$ exists and is locally compact, then $\prod_{j \in J}^C G_j$ has a compact open subgroup $\prod_{j \in J}^K H_j$; where each H_j is an open subgroup of G_j .*

Proof. Lemma 7 says that $\prod_{j \in J}^C G_j$ is a locally compact group with compact component and so $\prod_{j \in J}^C G_j$ has a compact open subgroup C . Putting $H_j = \text{pr}_j(C)$, where pr_j is the canonical map of $\prod_{j \in J}^C G_j$ into G_j , we have that H_j is in K and by Lemma 4, $C \subseteq \prod_{j \in J}^K H_j \subseteq \prod_{j \in J}^C G_j$. So $\prod_{j \in J}^K H_j$ is a compact open subgroup of $\prod_{j \in J}^C G_j$. By Proposition 1 and Lemma 1, pr_j

is a retraction and thus a quotient mapping. So each H_j is an open subgroup of G_j .

LEMMA 9. Under the hypotheses of Lemma 7 if $\prod_{j \in J}^C G_j$ exists and is locally compact, there exists a finite subset F of J such that for $j \in J \setminus F$, the group G_j has a compact open normal subgroup H_j with H_j a maximal compact subgroup of G_j .

Proof. Let H_j be as in Lemma 8. Let F_1 be the subset of J consisting of those j for which there is a compact group C_j with $H_j \leq C_j \leq G_j$. For $j \in J \setminus F_1$, set $C_j = H_j$. Then

$$\prod_{j \in J}^K H_j \leq \prod_{j \in J}^K C_j \leq \prod_{j \in J}^C G_j. \text{ So } \prod_{j \in J}^K H_j \text{ is an open subgroup of } \prod_{j \in J}^C G_j.$$

By Lemma 5, F_1 is finite.

Now set $F_2 = \{j \in J : H_j \text{ is not normal in } G_j\}$. So for each $j \in F_2$, there exists a $g_j \in G_j$ such that $H_j \neq g_j H_j g_j^{-1}$. For $j \in J \setminus F_2$, set $g_j = 1$. Define an automorphism α of $\prod_{j \in J}^C G_j$ by the

diagram

$$\begin{array}{ccc} G_j & \xleftarrow{\text{pr}_j} & \prod_{j \in J}^C G_j \\ I_{g_j} \downarrow & & \downarrow \alpha \\ G_j & \xleftarrow{\text{pr}_j} & \prod_{j \in J}^C G_j \end{array}$$

where $I_{g_j}(x) = g_j x g_j^{-1}$ for $x \in G_j$, and pr_j is the projection. Then

$\alpha\left(\prod_{j \in J}^K H_j\right)$ is open in $\prod_{j \in J}^C G_j$. But

$$\text{pr}_j \alpha\left(\prod_{j \in J}^K H_j\right) = I_{g_j} \text{pr}_j\left(\prod_{j \in J}^K H_j\right) = g_j H_j g_j^{-1}.$$

So

$$\alpha\left(\prod_{j \in J}^K H_j\right) \leq \prod_{j \in J}^K \left(g_j^H g_j^{-1}\right) \leq \prod_{j \in J}^C G_j ,$$

and thus $\prod_{j \in J}^K \left(g_j^H g_j^{-1}\right)$ is open in $\prod_{j \in J}^C G_j$. Hence

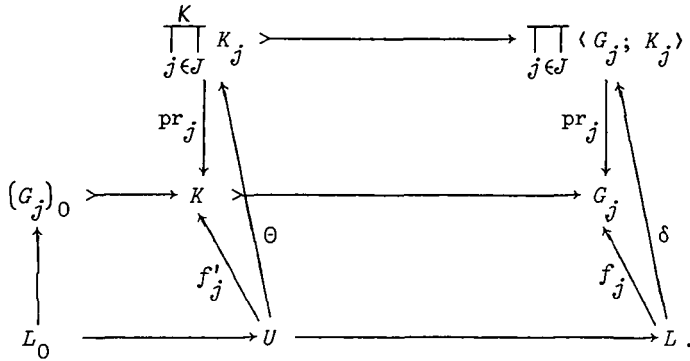
$$\left(\prod_{j \in J}^K H_j\right) \cap \left(\prod_{j \in J}^K g_j^H g_j^{-1}\right) = \prod_{j \in J}^K \left(H_j \cap g_j^H g_j^{-1}\right)$$

is open in both $\prod_{j \in J}^K H_j$ and $\prod_{j \in J}^K g_j^H g_j^{-1}$. By Lemma 5, $H_j \cap g_j^H g_j^{-1} = H_j$ for all but a finite number of j and also $H_j \cap g_j^H g_j^{-1} = g_j^H g_j^{-1}$ for all but a finite number of j . Hence F_2 is finite. The required set F is $F_1 \cup F_2$.

NOTATION. Let K_j be a compact open normal subgroup of a topological group G_j , for each $j \in J$. Let $\prod_{j \in J} K_j$ be the product of the K_j in *TopGps* . We denote by $\prod_{j \in J} \langle G_j; K_j \rangle$ the topological group which has as its underlying group the cartesian product $\prod_{j \in J} G_j$ and whose topology is determined by letting $\prod_{j \in J} K_j$, with its Tychonoff product topology, be an open subgroup. So $\prod_{j \in J} \langle G_j; K_j \rangle$ is a locally compact group.

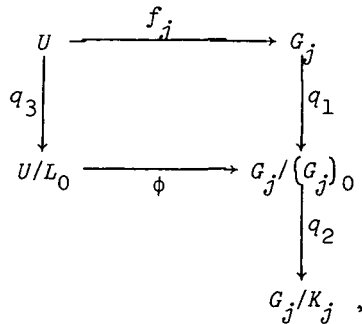
LEMMA 10. *Under the hypotheses of Lemma 7, if each K_j is a compact open normal subgroup of G_j such that G_j/K_j is torsion free and $\prod_{j \in J} \langle G_j; K_j \rangle$ is in C , then $\prod_{j \in J} \langle G_j; K_j \rangle$ is the product of $\{G_j; j \in J\}$ in L , the full subcategory of C whose objects are the locally compact C -objects.*

Proof.



Let L be any locally compact group in C and f_j a family of continuous homomorphisms of L into G_j . Let L_0 be the component of l and $(G_j)_0$ the component of G_j . As K_j is open in G_j , $(G_j)_0 \subseteq K_j$. Clearly $f_j(L_0) \subseteq (G_j)_0$. Let U be an open subgroup of L such that U/L_0 is compact.

Consider the diagram



where q_1, q_2 , and q_3 are the natural quotient maps. As $f_j(L_0) \subseteq (G_j)_0$, there is an induced continuous homomorphism $\phi : U/L_0 \rightarrow G_j/(G_j)_0$. Since U/L_0 is compact and G_j/K_j is torsion free and discrete, $q_2\phi(U/L_0) = \{1\}$. Hence $f'_j(U) \subseteq K_j$.

Let f'_j be the restriction of f_j to U . The family of maps

$f'_j : U \rightarrow K_j$ determine a continuous homomorphism $\Theta : U \rightarrow \prod_{j \in J}^K K_j$ such that $\text{pr}_j \Theta = f'_j$, for each $j \in J$. The maps f_j of L into G_j determine a (not necessarily continuous) homomorphism $\delta : L \rightarrow \prod_{j \in J} \langle G_j; K_j \rangle$ such that $\text{pr}_j \delta = f_j$, $j \in J$. Noting that $\delta|_U = \Theta$ and that Θ is continuous on the open subgroup U of L , we have that δ is continuous, and the proof of the lemma is complete.

THEOREM. *Let C be a full subcategory of TopGps with finite limits and K a full subcategory of C which is complete and has as its objects all compact subgroups of topological groups in C .*

(i) *Let $\{G_j; j \in J\}$ be a family in C such that $\prod_{j \in J}^C G_j$ exists and is locally compact. Then there exists a finite subset F of J such that for each $j \in J \setminus F$, the group G_j has a compact open normal subgroup K_j with G_j/K_j torsion free such that*

$$\prod_{j \in J}^C G_j \simeq \prod_{j \in F} G_j \times \prod_{j \in J \setminus F} \langle G_j; K_j \rangle$$

where $\prod_{j \in F}$ and \times denote the usual product in TopGps .

(ii) *Let L be the full subcategory of C having as its objects all locally compact groups in C . If $\{G_j; j \in J\}$ is a family of topological groups in L such that for each $j \in J \setminus F$, where F is a finite set, G_j has a compact open normal subgroup K_j with G_j/K_j torsion free, and if*

$\prod_{j \in J \setminus F} \langle G_j; K_j \rangle$ *is in L then*

$$\prod_{j \in F} G_j \times \prod_{j \in J \setminus F} \langle G_j; K_j \rangle$$

is the product in L of $\{G_j; j \in J\}$.

COROLLARY 1. *Let LC be the category consisting of all locally compact groups and all continuous homomorphisms between them. Let*

$\{G_j; j \in J\}$ be a family of topological groups in LC . Then $\prod_{j \in J}^{LC} G_j$ exists if and only if J has a finite subset F such that for each $j \in J \setminus F$, G_j has a compact open normal subgroup K_j such that G_j/K_j is torsion free.

COROLLARY 2. Let LCA be the category consisting of all locally compact abelian groups and all continuous homomorphisms between them. Let $\{G_j; j \in J\}$ be a family of topological groups in LCA . Then

$\prod_{j \in J}^{LCA} G_j$ exists if and only if J has a finite subset F , such that for each $j \in J \setminus F$, the group G_j is topologically isomorphic to $K_j \times D_j$, where K_j is a compact group and D_j is a discrete torsion free group.

Further if $\prod_{j \in J}^{LCA} G_j$ exists, then it is topologically isomorphic to

$\prod_{j \in F} G_j \times \prod_{j \in J \setminus F} K_j \times \prod_{j \in J \setminus F}^D D_j$, where \prod denotes the usual product in

TopGps and $\prod_{j \in J \setminus F}^D D_j$ denotes the cartesian product group $\prod_{j \in J \setminus F} D_j$ with the discrete topology.

By dualizing Corollary 2 we obtain

COROLLARY 3. Let LCA be the category consisting of all locally compact abelian groups and all continuous homomorphisms between them. Let $\{G_j; j \in J\}$ be a family of topological groups in LCA . Then the

coproduct $\prod_{j \in J}^{LCA} G_j$ exists if and only if J has a finite subset F such that for each $j \in J \setminus F$, the group G_j is topologically isomorphic to $A_j \times B_j$, where A_j is a discrete group and B_j is a compact connected

group. Further if $\prod_{j \in J}^{LCA} G_j$ exists, then it is topologically isomorphic to

$\prod_{j \in J} G_j \times \sum_{j \in J \setminus F}^D A_j \times \prod_{j \in J \setminus F}^K B_j$ where $\sum_{j \in J \setminus F}^D A_j$ denotes the restricted

direct sum of the A_j with the discrete topology and $\prod_{j \in J \setminus F}^K B_j$ the coproduct in the category of compact abelian groups of the B_j ; that is,

$$\prod_{j \in J \setminus F}^K B_j \text{ is } \left(\prod_{j \in J \setminus F}^D \hat{B}_j \right)^\wedge, \text{ where } \hat{} \text{ denotes the dual group.}$$

LEMMA 11. Let G be a closed subgroup of $K \times D$, where K is a compact abelian group and D is a discrete torsion free abelian group. Then G is topologically isomorphic to $K_\perp \times D_\perp$, where K_\perp is a compact group and D_\perp is a torsion free group.

Proof. By duality \hat{G} is a quotient group of $\hat{K} \times \hat{D}$, where \hat{K} is a discrete group and \hat{D} is a compact connected group. So \hat{G} is a quotient group of $F \times \hat{D}$, where F is a (suitable) discrete free abelian group. As \hat{D} is divisible, Proposition 5 of [1] implies that \hat{G} is topologically isomorphic to the product of a compact connected group and a discrete group. Dualizing again yields the required result.

COROLLARY 4. The largest full subcategory of LCA which contains the category of all compact abelian groups and has products (and coproducts) and kernels is the category whose objects are all those topological groups of the form $K \times D$, where K is a compact abelian group and D is a discrete torsion free abelian group.

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