Math. Nachr. 77, 177–180 (1977)

A Note on Free Topological Groupoids

By J. P. L. HARDY and Sidney A. MORRIS

(Eingegangen am 2. 1. 1975)

Introduction. In [11] MARKOV introduced the concept of a free topological group $F(X)$ on a topological space X and showed that, if X is any completely regular HAUSDORFF space, then $F(X)$ exists, is HAUSDORFF, and the canonical map $i\colon X \to F(X)$ is an embedding. His proof is rather long and tedious and since then a number of alternative have appeared – in particular we refer the reader to GRAEV [6], KAKUTANI [8], ŚWIERCZKOWSKI [16], THOMAS [17] and ORDMAN [15]. It was KAKUTANI who first observed that the actual existence of $F(X)$ is easily proved. Indeed since all that is required is a left adjoint to the forgetful functor from the category of topological groups to the category of topological spaces we only need to apply the FREYD special adjoint functor theorem. However it is still necessary to do some work to show that $F(X)$ is HAUSDORFF and $i\colon X \to F(X)$ is an embedding.

In this note we prove the more general result that for any completely regular HAUSDORFF topological graph Γ, the free topological groupoid $F(\Gamma)$ on Γ is HAUSDORFF and the canonical map $i\colon \Gamma \to F(\Gamma)$ is a topological graph embedding. (Other generalizations of a different nature have been investigated by MAL'CEV [10], ŚWIERCZKOWSKI [16] and MORRIS [12, 13, 14]. It should be noted that our proof depends heavily on the work of BROWN and HARDY [2]. They proved that for any k_ω-topological graph Γ, $F(\Gamma)$ is HAUSDORFF. (They did not show that $i\colon \Gamma \to F(\Gamma)$ is an embedding). Finally we record that our proof, even when specialized to the topological group case, yields a new proof of MARKOV's result.

Preliminaries. The theory of topological groupoids has been investigated in [3], [4] and [2], with the first of these having a bibliography of other papers in this area. We record here the definitions we require.

A *topological graph over* X consists of a topological space Γ of 'arrows', a topological space X of 'objects' and continuous functions ∂', $\partial\colon \Gamma \to X$, $u\colon X \to \Gamma$ called the initial, final and unit functions, respectively; these are to satisfy $\partial' u = \partial u = 1$. We usually confuse the graph with its space Γ of arrows and also write $X = \mathrm{Ob}(\Gamma)$.

The graph Γ becomes a *topological category* (over X) if there is also given a continuous composition $\vartheta\colon (a, b) \to ba$ with domain the set $\{(a, b) \in \Gamma \times \Gamma\colon \partial'(a) =$

$=\partial(b)\}$, making Γ into a category in the usual sense. Finally such a topological category is a *topological groupoid* if it is abstractly a groupoid (that is, a category with inverses, [7, 1]) and the inverse map $a \to a^{-1}$ is continuous. Morphisms of topological graphs, categories and groupoids are defined in the obvious way.

Let Γ be a topological graph. The *free topological groupoid on* Γ is a topological groupoid $F(\Gamma)$ together with a topological graph morphism $i\colon \Gamma \to F(\Gamma)$ such that if $f\colon \Gamma \to H$ is any topological graph morphism into a topological groupoid H then there is a unique topological groupoid morphism $f^*\colon F(\Gamma) \to H$ such that $f^* i = f$.

Once again by the FREYD special adjoint functor theorem [5] $F(\Gamma)$ exists for every topological graph Γ. Noting that every topological group is a topological groupoid with precisely one object, and that any topological space X with base point e defines a topological graph with arrows X, objects $\{e\}$ and $u\colon \{e\} \to X$ the inclusion map, we see that the free topological groupoid on the graph X is also the (GRAEV) free topological group on the space X [6].

Finally recall that the functor β which is the left adjoint to the forgetful functor from the category of compact HAUSDORFF spaces to the category of topological spaces is called the STONE-ČECH compactification [9]. If X is a topological space then the function $\beta\colon X \to \beta X$ is one-one if and only X is functionally separable. (A space X is said to be *functionally separable* if for each distinct pair of points x_1, x_2 of X there is a continuous function f from X into the real numbers with $f(x_1) \neq f(x_2)$.) Further $\beta\colon X \to \beta X$ is an embedding if and only if X is a completely regular HAUSDORFF space (see [9]).

Results

Proposition 1. Let Γ be any topological graph. Then there exists a compact HAUSDORFF topological graph Γ' (that is, the set of arrows of Γ' is a compact HAUSDORFF space) and a topological graph morphism $\gamma\colon \Gamma \to \Gamma'$ with the arrows of Γ' being $\beta\Gamma$ and $\mathrm{Ob}\,(\Gamma') = \beta\,\mathrm{Ob}\,(\Gamma)$.

Proof. As the STONE-ČECH compactification functor maps topological spaces into compact HAUSDORFF spaces, continuous functions into continuous functions, and the composite of two continuous functions into the composite of their images, it is clear that we can define Γ' to be the topological graph with arrows $\beta\Gamma$ objects $\beta\,\mathrm{Ob}\,\Gamma$ and structure functions $\beta\partial'$, $\beta\partial$ and βu, where ∂', ∂ and u are the structure functions of Γ. The morphism γ is the one induced by the map β from the space of arrows of Γ to the space of arrows of Γ'.

Remark. In the above Proposition, Γ' has the appropriate universal property; namely, that if f is any topological graph morphism of Γ into any compact HAUSDORFF topological graph Δ then there exists a unique topological graph morphism f^* of Γ' into Δ such that $f^*\gamma = f$. This also follows from the fact that β is a functor and is left adjoint to the forgetful functor from the category of compact HAUSDORFF spaces to the category of topological spaces.

Proposition 2. If Γ is any topological graph and $i\colon \Gamma \to F(\Gamma)$ is the canonical morphism from Γ to the free topological groupoid on Γ, then i is one-one (on the set of arrows) and $F(\Gamma)$ is algebraically the free groupoid on the abstract graph Γ.

Proof. Recall that if Γ is any abstract graph the free groupoid on Γ is characterized (up to isomorphism) as that groupoid F having a graph morphism $j\colon \Gamma \to F$ such that if f is any graph morphism of Γ into any groupoid G there exists a unique groupoid morphism $f^*\colon F \to G$ such that $f^*j = f$.

Let f be any graph morphism of Γ into a groupoid G. We make G into a topological groupoid by putting the indiscrete topology (that is, the topology in which the only open sets are the empty set and the space itself) on G. As any mapping into an indiscrete topological space is continuous, f is a topological graph morphism. Therefore there is a unique topological graph morphism f^* of $F(\Gamma)$ into G such that $f^*i = f$. As G has the indiscrete topology, f^* is also the unique graph morphism f^* of $F(\Gamma)$ into G such that $f^*i = f$. Therefore $F(\Gamma)$ is algebraically isomorphic to F, the free groupoid on the abstract graph Γ.

To verify that j is one-one consider the topological groupoid morphism f^* of $F(\Gamma)$ into F, with the indiscrete topology, induced by the topological graph morphism $j\colon \Gamma \to F$. As j is known to be one-one, f^*i is one-one. Hence i is one-one, as required.

Theorem. *Let Γ be any topological graph. Then $F(\Gamma)$ is functionally separable if and only if Γ is functionally separable. Further, if Γ is completely regular and* Hausdorff *then the canonical topological graph morphism $i\colon \Gamma \to F(\Gamma)$ is an embedding.*

Proof. Let $F(\Gamma)$ be functionally separable. Then as any topological space admitting a continuous one-one map into a functionally separable space is itself functionally separable, Proposition 3 implies that Γ is functionally separable.

Conversely assume Γ is functionally separable. By Proposition 1, there is a compact Hausdorff topological graph Γ' and a topological graph morphism $\gamma\colon \Gamma \to \Gamma'$. As the space of arrows of Γ' is $\beta\Gamma$ and Γ is functionally separable, we see γ is one-one. If i' is the canonical topological graph morphism $\Gamma' \to F(\Gamma')$ then $i'\gamma\colon \Gamma \to F(\Gamma')$ induces a topological groupoid morphism $\gamma^*\colon F(\Gamma) \to F(\Gamma')$, which by Proposition 2, and [1], 8.22. (Corollary 2), is also one-one. Thus if $F(\Gamma')$ is functionally separable $F(\Gamma)$ is functionally separable too.

Now Proposition 3 of [2] says that the free topological groupoid on any k_ω-topological graph is a k_ω-space. (A Hausdorff topological space is said to be a k_ω-space if it is a countable union of compact spaces and has the weak topology with respect to these subspaces.) So, in particular the free topological groupoid on a compact Hausdorff topological graph is functionally separable. Hence $F(\Gamma)$ and $F(\Gamma')$ are functionally separable.

Finally assume that Γ is completely regular and Hausdorff. Then $\gamma\colon \Gamma \to \Gamma'$ is an embedding. But as Γ' is compact Hausdorff and $F(\Gamma')$ is Hausdorff the one-one morphism $i': \Gamma' \to F(\Gamma')$ is also an embedding. So $i'\gamma$ is an embedding; that is, $\gamma^* i$ is an embedding. Hence i is an embedding, and the proof is complete.

Corollary. Let X be any topological space. Then the free topological group $F(X)$ on X is HAUSDORFF if and only if X is functionally separable. Further if X is completely regular and HAUSDORFF then the canonical map $i: X \to F(X)$ is an embedding.

Proof. This is an immediate consequence of our theorem once one observes that a topological group is a uniform space and is therefore completely regular, and that a completely regular space is HAUSDORFF if and only if it is functionally separable.

Acknowledgement. The research for this paper was done while the authors were at the University College of North Wales and were receiving financial assistance from the United Kingdom Science Research Council (S. R. C. Grant B/RG/3967).

Reference

[1] R. BROWN, Elements of Modern Topology (McGraw-Hill, 1968).
[2] R. BROWN and J. P. L. HARDY, Subgroups of free topological groups and free products of topological groups, J. London math. Soc. (2) **10**, 431–440 (1975).
[3] –, Topological groupoids: I Universal constructions, Diese Nachr. (to appear).
[4] R. BROWN, G. DANESH-NAURIE and J. P. L. HARDY, Topological groupoids: II covering morphisms and G-spaces, Diese Nachr. (to appear).
[5] P. FREYD, Abelian categories (New York, 1966).
[6] М. И. ГРАЕВ [M. I. GRAEV], Свободные топологические группы [Free topological groups], Izv. Akad. Nauk SSSR Ser. Mat. **12**, 279 – 324 (1948); Amer. math. Soc. Transl. **35**, (1951); reprinted Amer. math. Soc. Transl. (1) **8**, 305 – 364 (1962).
[7] P. J. HIGGINS, Notes on Categories and Groupoids, 1971.
[8] SHIZUO KAKUTANI, Free topological groups and infinite direct product topological groups, Proc. Imp. Acad. Tokyo **20**, 595 – 598 (1944).
[9] J. L. KELLEY, General topology, New York 1955.
[10] A. I. MAL'CEV, Free topological algebras, Izvestiya Akad. Nauk. SSSR Ser. Mat. **21**, 171 – 198 (1957), (Russian). English transl., Amer. math. Soc. Transl. (2) **17**, 173 – 200 (1961).
[11] A. A. МАРКОВ [A. A. MARKOV], О свободных топологических группах [On free topological groups], Izv. Acad. Nauk SSSR Ser. Math. **9**, 3 – 64 (1945); Amer. math. Soc. Transl. **30**, 11–88 (1950); reprinted Amer. math. Soc. Transl. (1) **8**, 195–272 (1962).
[12] S. A. MORRIS, Varieties of topological groups, Bull. Austral. math. Soc. **1**, 145 – 160 (1969).
[13] –, Varieties of topological groups II, Bull. Austral. math. Soc. **2**, 1–13 (1970).
[14] –, Locally compact groups and β-varieties of topological groups, Fundamenta Math. **78**, 23 – 26 (1973).
[15] E. T. ORDMAN, Free k-groups and free topological groups, General Topology and its Applications (to appear).
[16] S. ŚWIERCZKOWSKI, Topologies in free algebras, Proc. London math. Soc. (3) **14**, 566 – 576 (1964).
[17] B. V. S. THOMAS, Free topological groups, General Topology and its Applications **4**, 51 – 72 (1974).

Department of Mathematics,
St. Andrews University,
St. Andrews,
Scotland

School of Mathematics
University of New South Wales,
Kensington, N.S.W., 2033,
Australia