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THE FREE TOPOLOGICAL GROUP ON A SIMPLY CONNECTED SPACE

J. P. L. HARDY AND SIDNEY A. MORRIS

ABSTRACT. It is shown that the free k-group on a simply connected locally equiconnected space is simply connected. This result is then used to verify, for a large class of groups, the conjecture of Ordman that $\pi_1(G * H) = \pi_1(G) \times \pi_1(H)$, where $G * H$ is the free product of G and H.

1. **Preliminaries**. We begin by recalling a few definitions. A topological space X is said to be a *k-space* if a subset U of X is closed in X whenever $f^{-1}(U)$ is closed in C, for each compact Hausdorff space C and each continuous map $f\colon C \to X$. Further, X is said to be *k-Hausdorff* if $f(C)$ is closed in X, for each C and each f. Throughout the paper we will work in the category $\mathcal{K}$ of k-Hausdorff k-spaces and continuous maps. In particular, X will always denote a k-Hausdorff k-space.

We define a *k-group* G to be a group whose underlying space is a k-Hausdorff k-space and whose composition $\theta\colon G \times_k G \to G$ and inverse $\sigma\colon G \to G$ are continuous. ($G \times_k G$ denotes the product in $\mathcal{K}$ of two copies of G.) The *free k-group* [11], [7], [6] on a pointed space (X, e) consists of a k-group $F(X)$ and a continuous pointed map $i\colon X \to F(X)$ with the property that for any k-group G and any continuous pointed map f of X into G, there exists a unique continuous homomorphism f^* of $F(X)$ into G such that $f^*i = f$. It is proved in Ordman [11] and Hardy [7] that $F(X)$ always exists, is independent of the choice of base point, and is algebraically the free group on $i(X \setminus \{e\})$. Further, i maps X homeomorphically onto a closed subspace of $F(X)$; so we regard X itself as a closed subspace of $F(X)$.

Hardy [7, Chapter V, Theorem 3.1] shows that $F(X)$ is an iterated adjunction space [1]. (This can also be deduced from [11, §2] using [3, 4.5.8].) More precisely, if $F_n(X)$ denotes the closed subset of $F(X)$ comprising all the words of length at most n, then $F(X)$ has the weak topology with respect to the $F_n(X)$ and each $F_n(X)$ is an adjunction space as follows. Let X^{-1} be a homeomorphic copy of X with elements x^{-1} for each $x \in X$, and let $\overline{X}$ be the one-point union $X \vee X^{-1}$. Further, let $(\overline{X})^n$ denote the product in $\mathcal{K}$ of n copies of $\overline{X}$ and $p_n\colon (\overline{X})^n \to F_n(X)$ the map $(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n}) \to x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, where each $x_i \in X$ and $\varepsilon_i = \pm 1$. Then $F_n(X)$ is the adjunction spaces $F_{n-1}(X) \cup_{f_{n-1}} (\overline{X})^n$, where the attaching map f_{n-1} is the restriction of p_n to the closed subset $A_{n-1}(X) = p_n^{-1}(F_{n-1}(X))$ of $(\overline{X})^n$.

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For our theorem we will need to assume a mild homotopy extension condition. A space X is said to be *locally equiconnected* (LEC)[5], [4] if the inclusion of the diagonal $\Delta \subset X \times X$ is a closed cofibration. As examples we mention that every metric absolute neighbourhood retract (ANR) and every cell complex is LEC [5]. It is shown in [7] that if X is LEC then the inclusion $A_{n-1}(X) \subset (\bar{X})^n$ is a closed cofibration.

Our final recollection is that X is said to be *simply connected* [3] if each component of its fundamental groupoid is a tree groupoid [8]. Notice that a simply connected space need not be path connected.

2. Results.

THEOREM. *Let X be an LEC space. If X is simply connected then $F(X)$ is simply connected.*

PROOF. We shall prove by induction, that each $F_n(X)$ is simply connected. Since $F(X)$ has the weak topology with respect to the $F_n(X)$, the required result is then a trivial consequence of Lemma 9.3 of [13].

Clearly $F_1(X) = X \vee X^{-1}$ is simply connected. Assume $F_{n-1}(X)$ is simply connected. Consider the diagram where i and j are inclusions.

$$\begin{array}{ccc} A_{n-1}(X) & \xrightarrow{\;f_{n-1}\;} & F_{n-1}(X) \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle j} \\ (\bar{X})^n & \xrightarrow[\;p_n\;]{} & F_n(X) \end{array}$$

By our preliminary remarks this diagram is a pushout and the inclusion $A_{n-1}(X) \subset F_n(X)$ is a closed cofibration. So we can now employ the full strength of the Brown-van Kampen theorem [2], [3]. In so doing we must find "representative" subsets B, C and D of $F_n(X)$, $(\bar{X})^n$ and $F_{n-1}(X)$, respectively; the set C is also required to be representative in $A_{n-1}(X)$. (A set A is said to be representative in Z if $A \cap Z$ meets each path component of Z.)

Let D consist of one point from each path component of $F_{n-1}(X)$. Since f_{n-1} is onto we can choose C' to be a subset of $f_{n-1}^{-1}(D)$ comprising one point from each path component of $A_{n-1}(X)$. Finally let C be C' together with one point from each path component of $(\bar{X})^n$ not represented by C'.

In accordance with [3] we put $B = p_n(C)$ and so obtain that B, C and D are representative, as required. Thus the above square induces the following pushout diagram in the category of groupoids

$$\begin{array}{ccc} \pi(A_{n-1}(X), C) & \longrightarrow & \pi(F_{n-1}(X), D) \\ \Big\downarrow & & \Big\downarrow \\ \pi((\bar{X})^n, C) & \longrightarrow & \pi(F_n(X), B) \end{array}$$

(where $\pi(Z, A)$ denotes the restriction of the fundamental groupoid of Z to

A). To complete the proof we only have to show that $\pi(F_{n-1}(X), D)$, $\pi(A_{n-1}(X), C)$ and $\pi((\overline{X})^n, C)$ are discrete groupoids [8]; that is their only arrows are identity arrows. But we already have that the first two of these are discrete.

Of course $\pi((\overline{X})^n, C)$ is discrete if and only if C meets each path component of $(\overline{X})^n$ in precisely one point. Clearly this is the case if distinct path components of $A_{n-1}(X)$ lie in distinct path components of $(\overline{X})^n$. Indeed, while it is not in general true that the path components of a subspace are formed by intersecting that subspace with the path components of the whole space, this is so for the subspace $A_{n-1}(X)$ of $(\overline{X})^n$. Verifying this is rather tedious (but routine) so we include here only a typical case.

Consider $a \in A_{n-1}(X)$, where

$$a = \left(x_1^{\varepsilon_1}, \ldots, x_{i-3}^{\varepsilon_{i-3}}, x_{i-2}, x_{i-2}^{-1}, x_i, e, e, x_{i+3}^{\varepsilon_{i+3}}, e, x_{i+5}^{\varepsilon_{i+5}}, \ldots, x_n^{\varepsilon_n}\right),$$

with no x_j in the path component in $\overline{X}$ of e, and no $x_{j+1}^{\varepsilon_{j+1}}$ in the path component in $\overline{X}$ of $x_j^{-\varepsilon_j}$, but x_i and x_{i-2} in the same path component in $\overline{X}$. Let $C_j^{\varepsilon_j}$ denote the path component in $\overline{X}$ of $x_j^{\varepsilon_j}$, C_0 the path component in $\overline{X}$ of e and $\Delta_j^{\varepsilon_j}$ the set $C_j^{\varepsilon_j} \times C_j^{-\varepsilon_j} \cap \{(x, x^{-1}) \colon x \in \overline{X}\}$. Then the path component in $(\overline{X})^n$ of a is

$$\begin{aligned} C_a = {} & C_1^{\varepsilon_1} \times \cdots \times C_{i-3}^{\varepsilon_{i-3}} \times C_{i-2} \times C_{i-2}^{-1} \times C_{i-2} \\ & \times C_0 \times C_0 \times C_{i+3}^{\varepsilon_{i+3}} \times C_0 \times C_{i+5}^{\varepsilon_{i+5}} \times \cdots \times C_n^{\varepsilon_n} \end{aligned}$$

and clearly

$$\begin{aligned} C_a \cap A_{n-1}(X) = \Big[& C_1^{\varepsilon_1} \times \cdots \times C_{i-3}^{\varepsilon_{i-3}} \times \Delta_{i-2} \times C_{i-2} \\ & \times C_0 \times C_0 \times C_{i+3}^{\varepsilon_{i+3}} \times C_0 \times C_{i+5}^{\varepsilon_{i+5}} \times \cdots \times C_n^{\varepsilon_n} \\ & \cup\, C_1^{\varepsilon_1} \times \cdots \times C_{i-3}^{\varepsilon_{i-3}} \times C_{i-2} \times \Delta_{i-2}^{-1} \\ & \times C_0 \times C_0 \times C_{i+3}^{\varepsilon_{i+3}} \times C_0 \times C_{i+5}^{\varepsilon_{i+5}} \times \cdots \times C_n^{\varepsilon_n} \\ & \cup\, C_1^{\varepsilon_1} \times \cdots \times C_{i-3}^{\varepsilon_{i-3}} \times C_{i-2} \times C_{i-2}^{-1} \times C_{i-2} \\ & \times e \times C_0 \times C_{i+3}^{\varepsilon_{i+3}} \times C_0 \times C_{i+5}^{\varepsilon_{i+5}} \times \cdots \times C_n^{\varepsilon_n} \\ & \cup\, C_1^{\varepsilon_1} \times \cdots \times C_{i-3}^{\varepsilon_{i-3}} \times C_{i-2} \times C_{i-2}^{-1} \times C_{i-2} \\ & \times C_0 \times e \times C_{i+3}^{\varepsilon_{i+3}} \times C_0 \times C_{i+5}^{\varepsilon_{i+5}} \times \cdots \times C_n^{\varepsilon_n} \\ & \cup\, C_1^{\varepsilon_1} \times \cdots \times C_{i-3}^{\varepsilon_{i-3}} \times C_{i-2} \times C_{i-2}^{-1} \times C_{i-2} \\ & \times C_0 \times C_0 \times C_{i+3}^{\varepsilon_{i+3}} \times e \times C_{i+5}^{\varepsilon_{i+5}} \times \cdots \times C_n^{\varepsilon_n} \Big]. \end{aligned}$$

Since $\Delta_j^{\varepsilon_j}$ is homeomorphic to C_j, each of the terms in the above union is connected. Further as each of these terms also contains the point a, $C_a \cap A_{n-1}(X)$ is connected. Therefore $C_a \cap A_{n-1}(X)$ is the component in $A_{n-1}(X)$ of a, as required.

In [10] Ordman showed that if G and H are arbitrary k-groups, then $\pi_1(G * H) = \pi_1(G) \times \pi_1(H) \times L$, where $G * H$ denotes the coproduct in the category of k-groups, and L is some (unknown) group. He conjectured that L

is always trivial and showed that this is indeed true if G and H are countable CW-complexes with only one zero cell. In [9] and [12] it is shown that $G * H$ is homeomorphic to $G \times_k H \times_k F(G \wedge H)$, where $G \wedge H$ is the smash product of G and H; that is, $G \wedge H$ is the quotient space $G \times H/G \vee H$. So $\pi_1(G * H) = \pi_1(G) \times \pi_1(H) \times \pi_1(F(G \wedge H))$. Thus Ordman's problem is reduced to that of showing $\pi_1(F(G \wedge H))$ is always zero.

COROLLARY. *If X and Y are path connected LEC k-Hausdorff k-spaces then $\pi_1(F(X \wedge Y)) = 0$. Hence, if G and H are path connected LEC k-groups, then $\pi_1(G * H) = \pi_1(G) \times \pi_1(H)$.*

PROOF We prove $X \wedge Y$ is simply connected and LEC, and then apply our theorem. Simple connectedness and LEC are respectively obtained from (1) and (2) by taking $Z = X \times Y$ and $A = X \vee Y$:

Let Z and A be LEC k-Hausdorff k-spaces with A a closed subspace of Z.

(1) If Z and A are path connected and the induced homomorphism $\pi_1(A, a_0) \to \pi_1(Z, a_0)$ is surjective then the quotient space Z/A is simply connected.

(2) If the inclusion $A \subset Z$ is a closed cofibration then Z/A is LEC.

(Observe that $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$ and so $\pi_1(X \vee Y, a_0) \to \pi_1(X \times Y, a_0)$ is surjective. Also $X \vee Y \to X \times Y$ is a closed cofibration, cf. [1, Theorem 2.7, p. 234].)

(1) is easily obtained from the Brown-van Kampen theorem and the proof of (2) is routine.

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School of Mathematics and Computer Science, University College of North Wales, Bangor Ll57 2UW, Wales

Current address (J. P. L. Hardy): 6 Gwyllt Road, Llanfairfechan, LL33 0EE, Gwynedd, United Kingdom

Current address (S. A. Morris): Department of Pure Mathematics, University of New South Wales, Kensington, N.S.W., Australia