COLLOQUIUM MATHEMATICUM

VOL. XXXVI 1976 FASC. 2

THE VARIETY OF TOPOLOGICAL GROUPS GENERATED BY THE FREE TOPOLOGICAL GROUP ON [0, 1]

BY

SIDNEY A. MORRIS (BUNDOORA, VICTORIA)

It is shown* that the variety of topological groups generated by $F[0,1]$, the Graev free topological group on the unit interval $[0,1]$, contains all connected locally compact groups and all compact groups. However, the variety of topological groups generated by the class of all locally compact groups does not contain $F[0,1]$.

We assume that the reader is familiar with Graev free topological groups (see [5], [15], [7], [8], [17]) and varieties of topological groups (see [10]-[15]). We shall use the basic theorem (and notation of that theorem) on generating varieties of topological groups (see [1] and [3]).

LEMMA. *Let H be a Hausdorff topological group generated algebraically by a compact symmetric neighbourhood K of the identity. Then the canonical homomorphism f of $F(K)$ onto H is a quotient mapping.*

Proof. Let S be a subset of H such that $f^{-1}(S)$ is closed in $F(K)$. We are required to show that S is closed in H.

Let x be any point in the closure of S. Then there is a net $\{s_a\}$ in S converging to x. As Kx is a neighbourhood of x, $s_a \in Kx$ for sufficiently large a. Without loss of generality we can assume that this is true for all a. As K generates H, $x \in K^n$ for some n. So $s_a \in K^{n+1}$ for all a.

If $F_{n+1}(K)$ denotes the set of words in $F(K)$ of length not greater than $n+1$, with respect to K, then $f(F_{n+1}(K)) = K^{n+1}$. So, for each a, there is a $t_a \in F_{n+1}(K)$ with $f(t_a) = s_a$. Noting that $F_{n+1}(K)$ is compact, we see the net $\{t_a\}$ must have a convergent subnet $\{t_\delta\}$, where t_δ converges to y. As each t_δ is in the closed set $f^{-1}(S)$, $y \in f^{-1}(S)$. Now $f(t_\delta)$ converges to $f(y)$. But, as $f(t_\delta) = s_\delta$, and the net $\{s_a\}$ converges to x, we infer that $x = f(y)$. Hence $x \in S$, as required.

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* This research was done while the author was a United Kingdom Science Research Council Senior Visiting Fellow at the University College of North Wales.

THEOREM 1. *If V is any variety of topological groups containing $F[0,1]$, then V contains every locally compact group G which has quotient group $G/C(G)$ compact, where $C(G)$ is the component of the identity in G.*

Proof. According to the main approximation theorem for locally compact groups (see Section 4.6 of [9]), G is topologically isomorphic to a subgroup of a product $\prod_{i\in I} H_i$, where each H_i is a Lie group and a quotient group of G. As any connected locally compact group is compactly generated, 5.39 (i) of [6] yields that G is compactly generated. So each H_i, being a quotient of G, is also compactly generated. As V is closed under the formation of subgroups and products, it suffices to show that V contains every compactly generated Lie group H.

Let K be a symmetric compact neighbourhood of the identity in H such that K generates H algebraically. Then K is a finite-dimensional compact metric space. Further, by the Lemma, H is a quotient group of $F(K)$. As V is closed under the formation of quotients, we only have to show that V contains the free topological group on every finite-dimensional compact metric space. But Nickolas [17] has proved that $F[0,1]$ has $F(K)$ as a subgroup for every finite-dimensional compact metric space K. This completes the proof.

COROLLARY. *The variety of topological groups generated by $F[0,1]$ contains all connected locally compact groups and all compact groups.*

Remark. The variety of topological groups generated by $F[0,1]$ does not contain all locally compact groups. Indeed, it is shown in [12] that the variety of topological groups generated by a topological group of cardinality $\mathfrak{m}$ does not contain any discrete topological group of cardinality strictly greater than $\mathfrak{m}$. On the other hand, a very reasonable question is: Does the variety of topological groups generated by $F[0,1]$ contain every compactly generated locally compact group? (**P 990**) We do not know the answer. However, a similar (but simpler) argument to that in Theorem 1 shows that the variety of topological groups generated by $A[0,1]$, the Graev free abelian topological group, contains the topological group R of real numbers, and hence also contains every compactly generated locally compact abelian group.

THEOREM 2. *The variety of topological groups generated by the class of all locally compact groups does not contain $A[0,1]$.*

Proof. Suppose that $A[0,1]$ is in the variety generated by the class $\mathscr{L}$ of all locally compact groups. The basic theorem on generating varieties (see [1]) then says that

$$A[0,1]\in SC\bar{Q}\bar{S}P(\mathscr{L}) = SC(\mathscr{L});$$

that is,

$$A[0,1]\leqslant \prod_{i\in I} L_i,$$

where each L_i is a locally compact group and I is an index set. Let p_i be the projection of $A[0, 1]$ into L_i. As $A[0, 1]$ is abelian, so is the closure of $p_i(A[0, 1])$ in L_i, since $A[0, 1]$ is a connected locally compact abelian group. So we infer that

$$A[0, 1] \leqslant \prod_{i \in I} B_i,$$

where each B_i is a connected locally compact abelian group.

Note that Theorem 9.14 of [6] says that every connected locally compact abelian group is topologically isomorphic to $R^n \times K$ for some compact abelian group K and some non-negative integer n. It is shown in [7] that any k_ω-group, in particular $A[0, 1]$, is complete. So $A[0, 1]$ is a closed connected subgroup of a product of copies of R and a compact abelian group. Theorem 3 of [2] then implies that $A[0, 1]$ is topologically isomorphic to a product of copies R and a compact group. As $A[0, 1]$ does not contain a copy of the group of real numbers, this means that $A[0, 1]$ is compact. However, this is false (see [4] and [17]). Hence $A[0, 1]$ is not in the variety generated by the class of all locally compact groups.

COROLLARY. *$F[0, 1]$ is not in the variety of topological groups generated by the class of all locally compact groups.*

Proof. Simply note that $A[0, 1]$ is a quotient of $F[0, 1]$, and so $F[0, 1]$ cannot be in a variety unless $A[0, 1]$ is too.

Remark. It is shown in [17] that, for any non-totally path-disconnected space X, the Graev free topological group $F(X)$ on X contains $F[0, 1]$ as a subgroup. So Theorem 1 and the above Corollary remain true if $F[0, 1]$ is replaced by $F(X)$.

REFERENCES

[1] M. S. Brooks, S. A. Morris and S. A. Saxon, *Generating varieties of topological groups*, Proceedings of the Edinburgh Mathematical Society 18 (1973), p. 191-197.

[2] R. Brown, P. J. Higgins and S. A. Morris, *Countable products and sums of lines and circles; their subgroups, quotients and duality properties*, Mathematical Proceedings of the Cambridge Philosophical Society 78 (1975), p. 19-32.

[3] Su-shing Chen and S. A. Morris, *Varieties of topological groups generated by Lie groups*, Proceedings of the Edinburgh Mathematical Society 18 (1972), p. 49-53.

[4] R. M. Dudley, *Continuity of homomorphisms*, Duke Mathematical Journal 28 (1961), p. 587-594.

[5] М. И. Граев, *Свободные топологические группы*, Известия Академии наук СССР, серия математическая, 12 (1948), p. 279-324. English translation: American Mathematical Society Translations 35 (1951). Reprint: ibidem (1) 8 (1962), p. 305-364.

[6] E. Hewitt and K. A. Ross, *Abstract harmonic analysis I*, Berlin 1963.

[7] D. C. Hunt and S. A. Morris, *Free subgroups of free topological groups*, Proceedings of the 2nd International Group Theory Conference (Canberra 1973), Springer Lecture Notes 372, p. 377-388.

[8] J. Mack, S. A. Morris and E. T. Ordman, *Free topological groups and the projective dimension of a locally compact abelian group*, Proceedings of the American Mathematical Society 40 (1973), p. 303-308.

[9] D. Montgomery and L. Zippin, *Topological transformation groups*, New York - London 1955.

[10] S. A. Morris, *Varieties of topological groups*, Bulletin of the Australian Mathematical Society 1 (1969), p. 145-160.

[11] — *Varieties of topological groups II*, ibidem 2 (1970), p. 1-13.

[12] — *Varieties of topological groups III*, ibidem 2 (1970), p. 165-178.

[13] — *Varieties of topological groups generated by solvable and nilpotent groups*, Colloquium Mathematicum 27 (1973), p. 211-213.

[14] — *Locally compact groups and β-varieties of topological groups*, Fundamenta Mathematicae 58 (1973), p. 23-25.

[15] — *Varieties of topological groups and left adjoint functors*, Journal of the Australian Mathematical Society 16 (1973), p. 220-227.

[16] — and P. Nickolas, *Locally compact group topologies on algebraic free products of groups*, Journal of Algebra 38 (1976), p. 393-397.

[17] P. Nickolas, *Subgroups of the free topological group on* [0, 1], The Journal of the London Mathematical Society 12 (1976), p. 199-205.

DEPARTMENT OF MATHEMATICS
LA TROBE UNIVERSITY
BUNDOORA, VICTORIA
AUSTRALIA

Reçu par la Rédaction le 17. 4. 1975