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APPLICATIONS OF THE STONE-ČECH COMPACTIFICATION TO FREE TOPOLOGICAL GROUPS

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ABSTRACT. In this note the Stone-Čech compactification is used to produce short proofs of two theorems on the structure of free topological groups. The first is: *The free topological group on any Tychonoff space X contains, as a closed subspace, a homeomorphic copy of the product space X^n.* This is a generalization of a result of B. V. S. Thomas. The second theorem proved is C. Joiner's, Fundamental Lemma.

1. Introduction.

DEFINITION. Let X be any topological space. Then the compact Hausdorff space βX is said to be the *Stone-Čech compactification of* X if there exists a continuous map $\beta: X \to \beta X$ such that for any continuous map of X into any compact Hausdorff space K there exists a unique continuous map $\Phi: \beta X \to K$ such that $\Phi\beta = \phi$.

While βX exists and is unique for any topological space X, it is of particular interest when X is a Tychonoff (= completely regular Hausdorff) space, for then β is an embedding of X in βX and we can consider X to be a subspace of βX. (For details, see Kelley [5].)

DEFINITION. Let X be any Tychonoff space. Then the Hausdorff topological group $F(X)$ is said to be the *free topological group on* X if X is a subspace of $F(X)$ and for any continuous map ϕ of X into any topological group G there exists a unique continuous homomorphism $\Phi: F(X) \to G$ such that $\Phi|X = \phi$.

It is known [8], [1], [2] that for any Tychonoff space X, $F(X)$ exists and is unique. Considered as an abstract group $F(X)$ is the free group on the set X.

If, in the above definition, ' group ' is replaced everywhere by ' abelian group ' then we have the definition of the *free abelian topological group* $A(X)$ *on* X. Of course, considered as an abstract group, $A(X)$ is just the free abelian group on the set X.

Category theorists will recognize that if Top is the category of topological spaces and continuous maps and Comp. Haus. is the category of compact Hausdorff spaces and continuous maps, then the functor β: Top $\to$ Comp. Haus. is the left adjoint to the forgetful functor from Comp. Haus. to Top. The existence and uniqueness of $\beta(X)$ then follows from the Freyd adjoint functor theorem. Similar remarks can be made about $F(X)$ and $A(X)$. (For further details see Mac Lane [7].)

We prove here two theorems:

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THEOREM A. *Let X be any Tychonoff space. Then $F(X)$ and $A(X)$ both contain, as a closed subspace, a homeomorphic copy of the product space $X^n = X \times X \times \cdots \times X$, for each $n \geqq 1$.*

THEOREM B. *Let X be any Tychonoff space and let $F_n(X)$ $(A_n(X))$ denote the subspace of $F(X)$ $(A(X))$ comprising all the words of length $\leqq n$ with respect to X. Let $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$ be any reduced word, with $x_i \in X$ and $\varepsilon_i = \pm 1$. Then a base of neighbourhoods of $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$ in the subspace $F_n(X)$ $(A_n(X))$ is formed by the family of all subsets of the form $U_1^{\varepsilon_1} \cdots U_n^{\varepsilon_n}$ where U_i is a neighbourhood of x_i in X for $i = 1, \ldots, n$.*

B. V. S. Thomas [12] proved Theorem A for $A(X)$ only. Her proof cannot easily be extended to deal with $F(X)$. C. Joiner [4] proved Theorem B. In both cases, particularly the latter, the original proofs are significantly longer and more difficult than ours. Both of these papers include applications of the results. We do not give any further applications except to point out that Thomas' result can be derived easily from Joiner's. Further discussion of both results will be given in Nickolas [10].

2. **Proofs.** To introduce the essential idea of our Stone-Čech compactification proofs we begin with a well-known proposition. The proof we give here was first used in Morris [9, Theorem 2.2] for a related result. All other proofs known to us (for example, see [8], [1] and [2]) are much longer.

PROPOSITION. *Let X be any Tychonoff space. Then X and $F_n(X)$, for all $n \geqq 1$, are closed subspaces of $F(X)$.*

PROOF. Consider the free topological group $F(\beta X)$ on the Stone-Čech compactification βX of X. The natural map $\beta: X \to \beta X \subseteq F(\beta X)$ can, by the properties of free topological groups, be extended to a continuous homomorphism $j: F(X) \to F(\beta X)$. Clearly j is an algebraic isomorphism of $F(X)$ onto $j(F(X)) \subseteq F(\beta X)$. Noting that $F(X)$ is algebraically the free group on the set X and $F(\beta X)$ is algebraically the free group on the set βX, it is obvious that $j^{-1}(\beta X) = X$ and $j^{-1}(F_n(\beta X)) = F_n(X)$. Since βX and

$$F_n(\beta X) = \psi((\beta X \cup (\beta X)^{-1} \cup \{e\})^n),$$

where ψ denotes multiplication in $F(\beta X)$ and e denotes the identity in $F(\beta X)$, are compact, they are closed subspaces of the Hausdorff group $F(\beta X)$. From the continuity of j it then follows immediately that X and $F_n(X)$, $n \geqslant 1$, are closed subspaces of $F(X)$ as required.

REMARK. The essential feature of the above proof is that the natural map $\beta: X \to \beta(X) \subseteq F(\beta X)$ can be extended to a continuous algebraic isomorphism j of $F(X)$ onto its image in $F(\beta X)$.

PROOF OF THEOREM A. Consider the diagram

$$\begin{array}{ccc} (\beta X)^n & \xrightarrow{\Psi} & F(\beta X) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ X^n & \xrightarrow[\Phi]{} & F(X) \end{array}$$

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where X^n and $(\beta X)^n$ denote the product of n copies of X and of βX, respectively; i is the natural embedding of X^n in $(\beta X)^n$ given by $i(x_1, \dots, x_n) = (\beta(x_1), \dots, \beta(x_n))$; j is the map described in the above Remark; and Φ and Ψ are given by $(x_1, \dots, x_n) \to x_1 x_2^2 \cdots x_n^{2^{n-1}}$.

Clearly Φ and Ψ are continuous injections. Noting that $F(\beta X)$ is Hausdorff and $(\beta X)^n$ is compact, we see that Ψ is an embedding of $(\beta X)^n$ in $F(\beta X)$. Further, as $\Psi((\beta X)^n)$ is compact, it is a closed subspace of $F(\beta X)$. (So we have proved the theorem in the case when X is compact; that is, when $X = \beta X$.)

Since $\beta\colon X \to \beta(X)$ is an embedding, $i\colon X^n \to (\beta X)^n$ is also an embedding. Thus ψi is an embedding of X^n in $F(\beta X)$. Noting that the above diagram is commutative, this says that $j\Phi$ is an embedding of X^n in $jF(\beta X)$. Hence, Φ must be an embedding of X^n in $F(X)$. Finally, noting that $\Phi(X^n) = j^{-1}(\psi((\beta X)^n))$ and that $\psi((\beta X)^n)$ is closed in $F(\beta X)$, we have that $\Phi(X^n)$ is closed in $F(X)$.

The abelian case can be proved in a similar fashion.

Proof of Theorem B. Let U be any neighbourhood in $F_n(X)$ of $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$. Then $U = U' \cap F_n(X)$, where U' is a neighbourhood in $F(X)$ of $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$. Since $F(X)$ is a topological group, there exist $U_1', \dots, U_n'$ such that U_i' is a neighbourhood in $F(X)$ of x_i, $i = 1, \dots, n$ and $U_1'^{\varepsilon_1} \cdots U_n'^{\varepsilon_n} \subseteq U'$. Put $U_i = U_i' \cap X$. Then $U_1^{\varepsilon_1} \cdots U_n^{\varepsilon_n} \subseteq F_n(X)$, and each U_i is a neighbourhood in X of x_i. Hence, $U_1^{\varepsilon_1} \cdots U_n^{\varepsilon_n} \subseteq U' \cap F_n(X) = U$.

To complete the proof we have to show that every set of the form $U_1^{\varepsilon_1} \cdots U_n^{\varepsilon_n}$, where U_i is a neighbourhood of x_i in X, is a neighbourhood in $F_n(X)$ of $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$. It clearly suffices to show that every set of the form $U_1^{\varepsilon_1} \cdots U_n^{\varepsilon_n}$ is a neighbourhood in $F_n(X)$ of $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, where each $U_i = \beta^{-1}(V_i)$ and V_i is an open neighbourhood in βX of $\beta(x_i)$ such that $V_i = V_j$ if $x_i = x_j$ and $V_i \cap V_j = \varnothing$ if $x_i \neq x_j$.

Let Y be the subset of $F(\beta X)$ consisting of $\beta X \cup (\beta X)^{-1} \cup \{e\}$. Clearly these three sets are disjoint and open in Y. Similarly define Z to be the subset of $F(X)$ consisting of $X \cup X^{-1} \cup \{e\}$.

Consider the diagram

$$\begin{array}{ccc} Y^n & \xrightarrow{\ \psi\ } & F_n(\beta X) \\ \uparrow i & & \uparrow j \\ Z^n & \xrightarrow[\ \phi\ ]{} & F_n(X) \end{array}$$

where Y^n and Z^n are products of n copies of Y and of Z, respectively; i and j are the obvious maps; and ϕ and ψ are given by $(a_1, \dots, a_n) \to a_1 \cdot a_2 \cdots a_n$. Now $V_1^{\varepsilon_1} \times V_2^{\varepsilon_2} \times \cdots \times V_n^{\varepsilon_n}$ is an open subset of Y^n and $\psi((a_1, \dots, a_n)) \in V_1^{\varepsilon_1} \cdots V_n^{\varepsilon_n}$ if and only if $(a_1, \dots, a_n) \in V_1^{\varepsilon_1} \times \cdots \times V_n^{\varepsilon_n}$. Since ψ is surjective and $V_1^{\varepsilon_1} \times \cdots \times V_n^{\varepsilon_n}$ is an open saturated subset of the compact space Y^n, this implies $V_1^{\varepsilon_1} \cdots V_n^{\varepsilon_n}$ is an open subset of $F_n(\beta X)$. Since j is continuous and $j^{-1}(V_1^{\varepsilon_1} \cdots V_n^{\varepsilon_n}) = U_1^{\varepsilon_1} \cdots U_n^{\varepsilon_n}$, we have that $U_1^{\varepsilon_1} \cdots U_n^{\varepsilon_n}$ is an open neighbourhood in $F_n(X)$ of $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, which completes the proof for $F(X)$. The proof for $A(X)$ is similar.

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REMARK. We now point out that Theorem A can be deduced from Theorem B. To see this let X be any Tychonoff space, $F(X)$ its free topological group and $\Phi: X^n \to F(X)$ given by $\Phi((x_1, x_2, \ldots, x_n)) = x_1 x_2^2 \cdots x_n^{2^{n-1}}$. Clearly each word in $\Phi(X^n)$ has length precisely $m = 1 + 2 + \cdots + 2^{n-1}$. Since Φ is continuous and injective, we only have to show that $\Phi: X^n \to \Phi(X^n)$ is an open map. Let U be any neighbourhood of a point $(x_1, \ldots, x_n)$ in X^n. Then $U \supseteq U_1 \times U_2 \times \cdots \times U_n$ where U_i is a neighbourhood in X of x_i for each i. Then $\Phi(U) \supseteq \Phi(U_1 \times \cdots \times U_n) = U_1 U_2^2 \cdots U_n^{2^{n-1}}$, and by Theorem B this is a neighbourhood in $F_m(X)$, and hence also in $\Phi(X^n)$, of $\Phi((x_1, \ldots, x_n))$. So $\Phi: X^n \to \Phi(X^n)$ is an open map, as required.

We will conclude the paper with a question which arises quite naturally from our investigations.

Question. Let X be a Tychonoff space and j the canonical map from $F(X)$ to $F(\beta X)$. Under what conditions is j an embedding of (FX) in $F(\beta X)$?

We note that j is always continuous and injective. If X is compact then j is certainly an embedding since $X = \beta X$ and $F(X) = F(\beta X)$.

If X is a noncompact k_ω-space then j is not an embedding of $F(X)$ in $F(\beta X)$. (A topological space X is said to be a k_ω-space if $X = \bigcup_{n=1}^\infty X_n$, where each X_n is a compact Hausdorff space and X has the property that a subset A is closed in X if and only if $A \cap X_n$ is compact for each n. Note that the real line is an example of a k_ω-space. For further details, see [11].) If X is a k_ω-space then, by [6], $F(X)$ is a k_ω-space and, hence, by [3], it is a complete topological group. So if $j: F(X) \to F(\beta X)$ were an embedding, then (i) $j(F(X))$ would be the free topological group on $j(X)$, and (ii) $j(F(X))$ would be closed in $F(\beta X)$. By the Proposition this would imply that $j(X)$ is closed in $j(F(X))$ and hence also in $F(\beta X)$. But this says that $j(X) = \beta(X)$ is closed in βX–which is false since $\beta(X)$ is dense in βX.

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