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Countable products and sums of lines and circles: their closed subgroups, quotients and duality properties

By RONALD BROWN
University College of North Wales, Bangor, Gwynedd, Wales

PHILIP J. HIGGINS†
King's College, Strand, London, W.C. 2, England

AND SIDNEY A. MORRIS†
University of New South Wales, Kensington, N.S.W., Australia

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Introduction. It is well-known ((2), Theorem 9·11) that any closed subgroup of $\boldsymbol{R}^n$ is isomorphic (topologically and algebraically) to $\boldsymbol{R}^a \times \boldsymbol{Z}^b$, where a, b are suitable non-negative integers. For an infinite product of copies of $\boldsymbol{R}$, it is also known that any locally compact (hence closed) subgroup is a product of copies $\boldsymbol{R}$ and $\boldsymbol{Z}$, and that any connected subgroup is a product of copies of $\boldsymbol{R}$ (see (7), (3), respectively). Some information is also given in (3) on closed subgroups of products of copies of $\boldsymbol{R}$ and $\boldsymbol{T}$, where $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$ is the circle group.

In this paper, we study the class $\mathscr{D}_\Pi$ consisting of all Hausdorff Abelian groups topologically isomorphic to a product of a compact group with a countable product of copies of $\boldsymbol{R}$ and $\boldsymbol{Z}$. In §4, we prove:

THEOREM B. *Closed subgroups and Hausdorff quotients of groups in* $\mathscr{D}_\Pi$ *are again in* $\mathscr{D}_\Pi$.

Our proof relies heavily on Kaplan's extension (4, 5) of the Pontrjagin duality theorem; this extension allows us to dualise the problem. We therefore study also the class $\mathscr{D}_\Sigma$ consisting of all Hausdorff topological Abelian groups isomorphic to a sum of a discrete group with a countable sum of copies of $\boldsymbol{R}$ and $\boldsymbol{T}$. In §2, we prove:

THEOREM A. *Closed subgroups and Hausdorff quotients of groups in* $\mathscr{D}_\Sigma$ *are again in* $\mathscr{D}_\Sigma$.

Kaplan's results enable us to deduce Theorem B from Theorem A. They enable us also to formulate an extension of Pontrjagin duality as a functorial duality between the categories defined by $\mathscr{D}_\Pi$ and $\mathscr{D}_\Sigma$, taking closed inclusions and Hausdorff quotients to Hausdorff quotients and closed inclusions. Our methods are more elementary than those used in (12) to prove duality for ($\mathscr{L}_\infty$)-groups and we obtain specific information on the structure of subgroups and quotients. For example, Theorem B implies that

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any closed subgroup of a countable product of copies of $\boldsymbol{R}$ is a countable product of copies of $\boldsymbol{R}$ and $\boldsymbol{Z}$.

1. *Countable sums.* Let $\{H_i\}_{i=1}^{\infty}$ be a sequence of topological Abelian groups. (All topological groups considered will be assumed to be Hausdorff.) Their *direct sum* $\sum_{i=1}^{\infty} H_i$ is, algebraically, the subgroup of the product $\prod_{i=1}^{\infty} H_i$ consisting of elements (h_i) such that $h_i = 0$ for all but a finite number of i. The finite direct sums $\sum_{i=1}^{n} H_i$ are embedded in $\sum_{i=1}^{\infty} H_i$ in the obvious way. The topology we give to $\sum_{i=1}^{\infty} H_i$ is, in general, finer than that induced from the Tychonoff topology. It is the 'rectangular topology', defined in (4) as the topology induced from that topology on the product which has as a basis for its open sets all products ΠU_i of open sets U_i of H_i. It is easy to prove that with this topology $\sum_{i=1}^{\infty} H_i$ is a topological group, and clearly the induced topology on each finite sum $\sum_{i=1}^{n} H_i$ is the usual product topology.

PROPOSITION 1. *Let $\{H_i\}$ be a sequence of locally compact Abelian topological groups. Then a subset U of $\sum_{i=1}^{\infty} H_i$ is open if and only if U intersects each finite sum $\sum_{i=1}^{n} H_i$ in a relatively open set.*

Proof. Suppose that the subset U of $H = \sum_{i=1}^{\infty} H_i$ meets each $K_n = \sum_{i=1}^{n} H_i$ in a set open in K_n, and let $x \in U$. We construct, by induction, a rectangular neighbourhood of x contained in U. By definition of direct sum, $x = (x_1, x_2, \ldots, x_m, 0, 0, \ldots) \in K_m$ for some m. The set $U \cap K_m$ is therefore an open neighbourhood of x in K_m and contains a neighbourhood $D_m = C_1 \times C_2 \times \ldots \times C_m$ of x, where each C_i is a neighbourhood of x_i in H_i. Since H_i is locally compact, we can choose C_i to be compact. Suppose, inductively, that for some $n \geqslant m$, $C_1, C_2, \ldots, C_n$ are compact neighbourhoods of $x_1, x_2, \ldots, x_m, 0, \ldots, 0$ in $H_1, H_2, \ldots, H_m, \ldots, H_n$, respectively, such that

$$D_n = C_1 \times C_2 \times \ldots \times C_n \subset U \cap K_n.$$

Then the set $U \cap K_{n+1}$, open in K_{n+1}, contains the compact set D_n and so contains $D_n \times C_{n+1}$ for some neighbourhood C_{n+1} of 0 in H_{n+1}. Since H_{n+1} is locally compact, we may choose C_{n+1} to be compact, and this defines C_n for all n. The rectangular neighbourhood $H \cap (\Pi C_n)$ of x is clearly contained in U, and it follows that U is open in H. The converse is trivial.

COROLLARY. *If $H = \sum_{i=1}^{\infty} H_i$ and $K_n = \sum_{i=1}^{n} H_i$, where the H_i are locally compact Abelian groups, then in the category of topological Abelian groups, H is the direct limit of the chain of subgroups $K_1 \subset K_2 \subset \ldots$ and H is the coproduct of the subgroups H_i.*

Proof. It is clear that $H = \varinjlim K_n$ algebraically and the proposition shows that it is also the topological direct limit. For finite families in the category of topological

Abelian groups, product and coproduct are the same. Hence K_n is the coproduct of $H_1, H_2, \ldots, H_n$, and it follows that $H = \varinjlim K_n$ is the coproduct of all the H_i.

Remarks. 1. Proposition 4·3 of (8) claims that the coproduct of an arbitrary family of Abelian topological groups carries the 'asterisk' topology which, as shown in (4), agrees with the rectangular topology for countable families. However, the proof contains an error and the proposition is in fact false for an uncountable coproduct of copies of $\mathbf{R}$.

2. Proposition 1 and its Corollary are also related to Proposition 4 on p. 477 of (12).

Our chief applications of the direct sum will be to the cases when each H_i is the real line $\mathbf{R}$, the group of integers $\mathbf{Z}$ or the circle group $\mathbf{T}$. In particular, the countably infinite direct sum of copies $\mathbf{R}_i$ of $\mathbf{R}$ will be written $\mathbf{R}^\infty$; of copies $\mathbf{Z}_i$ of $\mathbf{Z}$ will be written $\mathbf{Z}^\infty$; and of copies $\mathbf{T}_i$ of $\mathbf{T}$ will be written $\mathbf{T}^\infty$. Notice that by Proposition 1, $\mathbf{Z}^\infty$ is a discrete topological group. Our immediate concern is with $\mathbf{R}^\infty$.

PROPOSITION 2.

(i) *Every finite-dimensional subspace F of $\mathbf{R}^\infty$ has the standard topology.*

(ii) *A subset of $\mathbf{R}^\infty$ is open if and only if it meets each finite-dimensional subspace F in an open subset of F.*

(iii) *$\mathbf{R}^\infty$ is a topological vector space.*

(iv) *Any linear mapping from $\mathbf{R}^\infty$ to a topological vector space is continuous.*

(v) *Any Hausdorff topological vector space V of algebraic dimension $\aleph_0$ over $\mathbf{R}$ and having property* (ii) *is isomorphic, as topological vector space, to $\mathbf{R}^\infty$.*

Proof. (i) Each $V_n = \mathbf{R}_1 \oplus \mathbf{R}_2 \oplus \ldots \oplus \mathbf{R}_n$ has the product topology which is the standard metric topology. Every finite-dimensional subspace F is a subspace of some V_n.

(ii) This follows immediately from Proposition 1.

(iii) $\mathbf{R}^\infty$ is a topological group, and we have to show that the scalar multiplication $\mathbf{R} \times \mathbf{R}^\infty \to \mathbf{R}^\infty$ is continuous. Now the topology on $\mathbf{R} \times \mathbf{R}^\infty$ is the same as the rectangular topology on $\mathbf{R} \oplus \mathbf{R}_1 \oplus \mathbf{R}_2 \oplus \ldots$. Therefore a subset of $\mathbf{R} \times \mathbf{R}^\infty$ is open if and only if it meets each $\mathbf{R} \times V_n$ in an open set. However, V_n has the standard topology and is a topological vector space. Thus the scalar multiplication is continuous on each $\mathbf{R} \times V_n$, and so is continuous.

(iv) By Proposition 1, a function on $\mathbf{R}^\infty$ is continuous if and only if its restriction to each finite sum $V_n \cong \mathbf{R}^n$ is continuous. But a linear mapping from $\mathbf{R}^n$ to a topological vector space is well-known to be continuous.

(v) Let V be a Hausdorff topological vector space of dimension $\aleph_0$ over $\mathbf{R}$ and choose a linear isomorphism $\theta: V \to \mathbf{R}^\infty$. Assume that V has property (ii). Since θ and θ^{-1} send finite-dimensional subspaces to finite-dimensional subspaces, it is sufficient to show that θ induces a homeomorphism between corresponding finite-dimensional subspaces. But this follows from a theorem of Tychonoff (11, 10) which asserts that a finite-dimensional vector space over $\mathbf{R}$ has only one Hausdorff topology which makes it a topological vector space.

COROLLARY 1. *Every vector subspace of $\mathbf{R}^\infty$ is closed.*

Proof. The complement of a vector subspace meets each finite subspace W in an open subset of W.

COROLLARY 2. *If $a_1, a_2, \ldots$ is any $\boldsymbol{R}$-basis for $\boldsymbol{R}^\infty$, and if W_n is the subspace spanned by $\{a_1, \ldots, a_n\}$, then $\boldsymbol{R}^\infty$ is the direct limit in the category of topological groups of the chain of subgroups $W_1 \subset W_2 \subset \ldots$.*

Proof. Since, by (iv), every linear automorphism of $\boldsymbol{R}^\infty$ is a homeomorphism, we need to prove this result only for the standard basis and the chain of subgroups $\boldsymbol{R}_1 \subset \boldsymbol{R}_1 \oplus \boldsymbol{R}_2 \subset \ldots$. But this is a special case of the corollary to Proposition 1.

COROLLARY 3. *If $a_1, a_2, \ldots$ is any $\boldsymbol{R}$-basis of $\boldsymbol{R}^\infty$, then $\boldsymbol{R}^\infty$ is the coproduct in the category of topological Abelian groups of the groups $\boldsymbol{R}a_i$, $i = 1, 2, \ldots$, and has the rectangular topology with respect to the decomposition $\boldsymbol{R}^\infty = \sum_{i=1}^{\infty} \boldsymbol{R}a_i$.*

Proof. This also follows from the corollary to Proposition 1.

COROLLARY 4. *If V is a vector subspace of $\boldsymbol{R}^\infty$, then V is topologically isomorphic to $\boldsymbol{R}^\infty$ or to $\boldsymbol{R}^n$ for some n, and $\boldsymbol{R}^\infty$ contains a vector subspace V' such that $\boldsymbol{R}^\infty$ is algebraically and topologically $V \oplus V'$.*

Proof. This follows from Corollary 3 since every $\boldsymbol{R}$-basis of a vector subspace can be extended to an $\boldsymbol{R}$-basis of $\boldsymbol{R}^\infty$.

Remark. We have shown that $\boldsymbol{R}^\infty$ is a topological vector space and that it carries the finest group topology consistent with the standard topology on its one-dimensional subspaces. The rectangular topology is clearly locally convex, so it is the finest locally convex topology. In other words, $\boldsymbol{R}^\infty$ is a totally fine space in the sense of Kaplan (6), and $\boldsymbol{R}^\infty$ coincides with the space φ of (9). Corollary 1 and results similar to Corollaries 2 and 3 were proved in (6) for totally fine spaces.

2. *Closed subgroups and quotients of direct sums.* In this section, we determine the structure of the closed subgroups and the Hausdorff quotients of certain direct sums of locally compact Abelian groups. The case of $\boldsymbol{R}^\infty$ is of special interest and our results extend the well-known description of closed subgroups and Hausdorff quotients of $\boldsymbol{R}^n$. In §4, we shall obtain a different extension of the classical results to countable *products* of copies of $\boldsymbol{R}$.

THEOREM 1. *Let B be a closed subgroup of $\boldsymbol{R}^\infty$. Then there is an $\boldsymbol{R}$-basis $\{x_i\}$, $i = 1, 2, \ldots$ for $\boldsymbol{R}^\infty$ such that $B = \sum_{i=1}^{\infty} B_i$, where B_i is a closed subgroup of $\boldsymbol{R}x_i$ for each i.*

The proof is given later.

COROLLARY. *Every closed subgroup of $\boldsymbol{R}^\infty$ has the form $\boldsymbol{R}^a \oplus \boldsymbol{Z}^b$, and every Hausdorff quotient group of $\boldsymbol{R}^\infty$ has the form $\boldsymbol{R}^c \oplus \boldsymbol{T}^b$, where a, b, c are non-negative integers or ∞.*

Proof. The description of closed subgroups in the first part of the corollary is immediate from the theorem; the rectangular topology on $\Sigma \boldsymbol{R}x_i$ (see Proposition 2, Corollary 3) induces the rectangular topology on $B = \Sigma B_i$. Since $\boldsymbol{R}^\infty$ and B are, respectively, the coproducts of the topological Abelian groups $\boldsymbol{R}x_i$ and B_i, the quotient group $\boldsymbol{R}^\infty/B$ is the coproduct of the groups $(\boldsymbol{R}x_i)/B$, each isomorphic to $\boldsymbol{R}$ or $\boldsymbol{T}$. Since these factors are locally compact, the topology on the countable coproduct

$$\boldsymbol{R}^\infty/B = \Sigma(\boldsymbol{R}x_i)/B_i$$

is the rectangular topology.

The proof of Theorem 1 follows from two propositions, the first of which generalizes Theorem 2 of ch. VII, section 1 of (1).

PROPOSITION 3. *Let B be a closed subgroup of $\boldsymbol{R}^\infty$. Then $\boldsymbol{R}^\infty$ can be written, algebraically and topologically, as the direct sum $U \oplus V \oplus W$ of vector subspaces such that*

(i) *U is the largest vector subspace of B;*

(ii) *$V \cap B$ is discrete and spans V;*

(iii) *$W \cap B = \{0\}$.*

Proof. Let U be the union of all one-dimensional subspaces contained in B. Then U is clearly a vector subspace. Let U' be any algebraically complementary subspace of U in $\boldsymbol{R}^\infty$. Then B is algebraically the direct sum of U and $U' \cap B$. Let V be the vector subspace spanned by $U' \cap B$, and let W be any complementary subspace of W in U'. Then $W \cap B = \{0\}$ and $\boldsymbol{R}^\infty = U \oplus V \oplus W$ algebraically, and also topologically; this last follows from Corollary 3 of Proposition 2 by taking an $\boldsymbol{R}$-basis for each of U, V and W.

Clearly (i) and (iii) are satisfied, and it remains to prove that $V \cap B$ is discrete.

For each finite-dimensional subspace F of $\boldsymbol{R}^\infty$, $V \cap B \cap F$ is a closed subgroup of F and contains no one-dimensional subspace. Since F has the standard topology (Proposition 2(i)), this implies that $V \cap B \cap F$ is discrete ((1) ch. VII, section 1, Proposition 3). Hence every subset of $V \cap B$ meets each finite-dimensional F in a closed subset of F and is therefore closed in $\boldsymbol{R}^\infty$ (as follows easily from Proposition 2(ii)). Therefore $V \cap B$ is discrete.

Our next task is to characterize the discrete subgroups of $\boldsymbol{R}^\infty$ (compare (1) ch. VII, section 1).

PROPOSITION 4. *Every discrete subgroup of $\boldsymbol{R}^\infty$ is topologically isomorphic to $\boldsymbol{Z}^\infty$, or to $\boldsymbol{Z}^n$ for some n, and has a $\boldsymbol{Z}$-basis which is linearly independent over $\boldsymbol{R}$.*

Proof. Let B be a discrete subgroup of $\boldsymbol{R}^\infty = \sum_{i=1}^{\infty} \boldsymbol{R}_i$. Let $V_n = \sum_{i=1}^{n} \boldsymbol{R}_i$ and let $B_n = B \cap V_n$. We will construct a sequence $a_1, a_2, \ldots$ (possibly finite) of elements of B such that (i) $a_1, a_2, \ldots$ are linearly independent in $\boldsymbol{R}^\infty$, and (ii) for some sequence of non-negative integers $i_1 \leqslant i_2 \leqslant \ldots$, the elements $a_1, a_2, \ldots, a_{i_n}$ generate the group B_n for each n. Since $B = \bigcup B_n$, it will then follow that $a_1, a_2, \ldots$ freely generate B as an Abelian group, whence B is isomorphic to one of the discrete groups $\boldsymbol{Z}^\infty$ or $\boldsymbol{Z}^n$ for some n.

The sequence is constructed by induction on n. First, B_1 is a discrete subgroup of $V_1 = \boldsymbol{R}_1$ and is therefore trivial or isomorphic to $\boldsymbol{Z}$. In the first case, take $i_1 = 0$, and in the second take $i_1 = 1$ and a_1 to be a generator for B_1.

Now suppose that we have constructed the sequence as far as a linearly independent set of generators $a_1, \ldots, a_r$ for $B_n = B \cap V_n$ (where $r = i_n$). The group B_{n+1} is a discrete subgroup of $V_{n+1} \cong \boldsymbol{R}^{n+1}$, so is isomorphic to $\boldsymbol{Z}^m$ for some $m \leqslant n+1$, and any $\boldsymbol{Z}$-basis for B_{n+1} is linearly independent over $\boldsymbol{R}$ (see (1) ch. 7, section 1, Theorem 1). To complete the proof, we must show that a $\boldsymbol{Z}$-basis for B_{n+1} exists which contains the elements $a_1, \ldots, a_r$ already constructed. Now B_{n+1} is a finitely generated Abelian group, and therefore so also is B_{n+1}/B_n. But

$$B_{n+1}/B_n = B_{n+1}/(V_n \cap B_{n+1}) \cong (V_n + B_{n+1})/V_n,$$

which is a subgroup of $V_{n+1}/V_n \cong \boldsymbol{R}$. Hence B_{n+1}/B_n is torsion-free and is therefore free Abelian of finite rank. If $b_1, \ldots, b_s$ is a $\boldsymbol{Z}$-basis for this group, and $a_{r+1}, \ldots, a_{r+s}$ are representatives of $b_1, \ldots, b_s$ in B_{n+1}, then $a_1, \ldots, a_{r+s}$ is a $\boldsymbol{Z}$-basis for B_{n+1} of the required form.

Proof of Theorem 1. We decompose $\boldsymbol{R}^\infty = U \oplus V \oplus W$ as in Proposition 3, choose a $\boldsymbol{Z}$-basis for $V \cap B$ which is an $\boldsymbol{R}$-basis for V, and adjoin to this any $\boldsymbol{R}$-basis for U and any $\boldsymbol{R}$-basis for W.

Our next proposition is needed in order to give a similar description of the closed subgroups and Hausdorff quotients of more general direct sums.

PROPOSITION 5. *Let $H = V \oplus F$, where V is a divisible Abelian topological group and F is a discrete free Abelian group. Let B be any closed subgroup of H. Then there is a discrete free Abelian subgroup F' of H, isomorphic to F, such that topologically and algebraically*

(i) $H = V \oplus F'$ *and* (ii) $B = (B \cap V) \oplus (B \cap F')$.

Proof. Let $\pi_1 : H \to V, \pi_2 : H \to F$ be the projections. The restriction of π_2 to B is a homomorphism from B to F with kernel $B \cap V$. Since F is free Abelian, it follows that $B/B \cap V$ is free Abelian, and so B splits algebraically as a direct sum

$$B = (B \cap V) \oplus C,$$

where C is a free Abelian subgroup of B. We look for a complement F' of V which contains C.

Let p_1, p_2 be the restrictions of π_1, π_2 to C. Then p_2 is an injection (since

$$C \cap V = C \cap B \cap V = \{0\})$$

and V, being divisible, is injective. Therefore there is a group homomorphism $\theta : F \to V$,

$$\begin{tikzcd} C \arrow[r, "p_2"] \arrow[d, "p_1"'] & F \arrow[dl, dashed, "\theta"] \\ V & \end{tikzcd}$$

such that $\theta \circ p_2 = p_1$. Putting $\phi = 1 + \theta : F \to H$ and $F' = \phi(F)$, we have $H = V \oplus F'$ algebraically, the decomposition being given by $v + f = (v - \theta(f)) + (f + \theta(f))$ for $v \in V$, $f \in F$. Also $C \subset F'$, since for c in C we have

$$c = p_1(c) + p_2(c) = \theta p_2(c) + p_2(c) = \phi p_2(c) \in \phi(F).$$

Thus (i) and (ii) are satisfied algebraically.

Now $\phi : F \to F'$ is an algebraic isomorphism and since ϕ^{-1} is induced by π_2, ϕ^{-1} is continuous. But F is discrete, so ϕ is a homeomorphism and F' is a discrete free Abelian group.

In order to show that H has the product topology with respect to the decomposition $H = V \oplus F'$, it is enough to show that the corresponding projections $\pi_1': H \to V$ and $\pi_2': H \to F'$ are continuous. But this is clearly the case, since $\pi_1' = \pi_1 - \theta \circ \pi_2$ and $\pi_2' = \pi_2 + \theta \circ \pi_2$. Hence the decomposition $B = (B \cap V) \oplus (B \cap F')$ also has the direct sum topology.

COROLLARY. *If F is a discrete free Abelian group, then any closed subgroup of $\boldsymbol{R}^\infty \oplus F$ is algebraically and topologically of the form $\boldsymbol{R}^a \oplus C$, where C is a discrete free Abelian group, and any Hausdorff quotient of $\boldsymbol{R}^\infty \oplus F$ is of the form $\boldsymbol{R}^b \oplus \boldsymbol{T}^c \oplus D$, where D is a discrete group and a, b, c are non-negative integers or ∞.*

Proof. Since $\boldsymbol{R}^\infty$ is divisible, we have, for any closed subgroup B of $\boldsymbol{R}^\infty \oplus F$, decompositions $\boldsymbol{R}^\infty \oplus F'$ and $B = (B \cap \boldsymbol{R}^\infty) \oplus (B \cap F')$, where F' is free and discrete. By Theorem 1, $B \cap \boldsymbol{R}^\infty$ is of the form $\boldsymbol{R}^a \oplus \boldsymbol{Z}^c$ and $\boldsymbol{R}^\infty/(B \cap \boldsymbol{R}^\infty)$ is of the form $\boldsymbol{R}^b \oplus \boldsymbol{T}^c$. Since $B \cap F'$ is a discrete free Abelian group and $F'/(B \cap F')$ is discrete, the corollary follows.

We now introduce a category $\mathscr{D}_\Sigma$, whose objects are all those topological Abelian groups which are sums of a discrete group with a countable (and so possibly finite) sum of copies of $\boldsymbol{R}$ and $\boldsymbol{T}$. The morphisms of $\mathscr{D}_\Sigma$ are the continuous homomorphisms. Our main result on sums is:

THEOREM A. *Every closed subgroup, and every Hausdorff quotient group, of an object of $\mathscr{D}_\Sigma$ is again in $\mathscr{D}_\Sigma$.*

Proof. Let $\boldsymbol{R}^b \oplus \boldsymbol{T}^c \oplus D$ be any group in $\mathscr{D}_\Sigma$, where D is discrete and b and c are non-negative integers or ∞. There exists a quotient morphism

$$p: \boldsymbol{R}^\infty \oplus F \to \boldsymbol{R}^b \oplus \boldsymbol{T}^c \oplus D,$$

where F is a discrete free group. Hence any Hausdorff quotient of $\boldsymbol{R}^b \oplus \boldsymbol{T}^c \oplus D$ is also a Hausdorff quotient of $\boldsymbol{R}^\infty \oplus F$ and so by the Corollary of Proposition 5, it is in $\mathscr{D}_\Sigma$. To complete the proof, let B be a closed subgroup of $\boldsymbol{R}^b \oplus \boldsymbol{T}^c \oplus D$. Then $p^{-1}(B)$ is a closed subgroup of $\boldsymbol{R}^\infty \oplus F$ and so, by the same corollary, $p^{-1}(B)$ is of the form $\boldsymbol{R}^d \oplus F_1$ where F_1 is discrete and free and d is a non-negative integer or ∞. Since B is a quotient of $p^{-1}(B)$, it follows that B lies in $\mathscr{D}_\Sigma$.

3. *Duality between sums and products.* By a *duality* between Abelian topological groups G and H, we mean a bi-additive pairing $\phi: G \times H \to \boldsymbol{T}$, continuous in each variable separately, such that the induced maps $\phi^G: G \to H^\wedge$ and $\phi^H: H \to G^\wedge$ are isomorphisms of topological groups. Here the character groups $G^\wedge$, $H^\wedge$ of G, H are, as usual, the groups of continuous homomorphisms into the circle group $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$ and they carry the compact-open topology.

In (4), Kaplan extended the Pontrjagin duality theorem to products and sums of dual groups. In this section, we state his main theorem and prove, in a form suitable for our purposes, various other results implicit in (4) and (5).

KAPLAN'S THEOREM. *Let G_λ, $H_\lambda (\lambda \in \Lambda)$ be Abelian topological groups and suppose that, for each $\lambda \in \Lambda$, $\phi_\lambda: G_\lambda \times H_\lambda \to \boldsymbol{T}$ is a duality. Let $G = \prod_{\lambda \in \Lambda} G_\lambda$ and $H = \sum_{\lambda \in \Lambda} H_\lambda$. Then the pairing $\phi: G \times H \to \boldsymbol{T}$ defined by $\phi(g, h) = \Sigma \phi_\lambda(g_\lambda, h_\lambda)$ is a duality.*

In this theorem, ΠG_λ is the usual product with the Tychonoff topology, and ΣH_λ is the direct sum with the 'asterisk' topology (see (4) for the definition of this). Kaplan shows that when the indexing set Λ is countable, the asterisk topology is the same as the rectangular topology which we have used in section 2. The only other fact we shall need about the asterisk topology, is that it is always at least as fine as the rectangular topology. Note that the sum $\Sigma\phi_\lambda(g_\lambda, h_\lambda)$ is finite because each $h = (h_\lambda)$ has only a finite number of non-zero components. We shall always consider the groups G_λ, H_λ as subgroups of ΠG_λ and ΣH_λ, namely, the subgroups consisting of all elements whose μth components are zeros for all $\mu \neq \lambda$; with respect to this embedding they carry the subgroup topology.

For any pairing $\phi: G \times H \to T$ and any subgroup A of G, the *annihilator of A in H* is the subgroup

$$A^0 = \{h \in H;\ \ \phi(a,h) = 0 \quad \text{for} \quad \text{all}\ a \in A\}.$$

Similarly, for $B \subset H$, $\quad B^0 = \{g \in G;\ \ \phi(g,b) = 0 \quad \text{for} \quad \text{all}\ b \in B\}.$

If ϕ is a duality, then $G^0 = \{0\}$ and $H^0 = \{0\}$. We shall say that an Abelian topological group C is *reflexive* if the natural pairing $C^\wedge \times C \to T$ is a duality, that is, if the canonical map $C \to C^{\wedge\wedge}$ is a topological isomorphism.

PROPOSITION 6. (i) *Let $G_\lambda, H_\lambda, \phi_\lambda, G, H$ and ϕ be as in Kaplan's theorem. Then $G_\lambda = (H'_\lambda)^0$ and $H_\lambda = (G'_\lambda)^0$, where $G'_\lambda = \prod_{\mu \neq \lambda} G_\mu$ and $H'_\lambda = \sum_{\mu \neq \lambda} H_\mu$.*

(ii) *Let $\phi: G \times H \to T$ be an arbitrary duality, and suppose that $H = \sum_{\lambda \in \Lambda} H_\lambda$, where the subgroups H_λ of H are reflexive. Put $G_\lambda = (\sum_{\mu \neq \lambda} H_\mu)^0$. Then $G = \prod_{\lambda \in \Lambda} G_\lambda$ and $\phi(g,h) = \Sigma\phi_\lambda(g_\lambda, h_\lambda)$, where ϕ_λ is the pairing $G_\lambda \times H_\lambda \to T$ induced by ϕ. Furthermore, each ϕ_λ is a duality.*

Proof. (i) $g \in (H'_\lambda)^0 \Leftrightarrow \phi(g,h) = 0$ whenever $h_\lambda = 0$

$\Leftrightarrow \phi_\mu(g_\mu, H_\mu) = 0$ for all $\mu \neq \lambda$.

But ϕ_μ is a duality, so the annihilator of H_μ in G_μ is trivial. Hence $g \in (H'_\mu)^0 \Leftrightarrow g_\mu = 0$ for all $\mu \neq \lambda$.

(ii) Let $K_\lambda = H^\wedge_\lambda$ and let $K = \prod_{\lambda \in \Lambda} K_\lambda$. The natural dualities $\psi_\lambda: K_\lambda \times H_\lambda \to T$ induce, by Kaplan's theorem, a duality $\psi: K \times H \to T$, with $\psi(k,h) = \Sigma\psi_\lambda(k_\lambda, h_\lambda)$. The dualities $\phi: G \times H \to T$ and $\psi: K \times H \to T$ induce topological isomorphisms $\phi^G: G \to H^\wedge$ and $K \to H^\wedge$, so there is a topological isomorphism $\theta: K \to G$ such that $\phi^G \circ \theta = \psi^K$, that is, $\phi(\theta(k), h) = \psi(k,h)$ for all $k \in K, h \in H$. Since, by (i), K_λ is the annihilator of H'_λ in K, its image under θ is precisely G_λ, and it follows that $G = \Pi G_\lambda$. Also

$$\phi(g,h) = \psi(\theta^{-1}(g), h) = \Sigma\psi_\lambda(\theta^{-1}(g)_\lambda, h_\lambda) = \Sigma\phi_\lambda(g_\lambda, h_\lambda).$$

Finally, since H_λ is reflexive, ψ_λ is a duality and it follows that ϕ_λ is also a duality.

PROPOSITION 7. *Let $\phi: G \times H \to T$ be a duality, let B be a closed subgroup of H and let $A = B^0$. Suppose that H has a decomposition $H = \Sigma H_\lambda$, where the H_λ are reflexive groups, such that $B = \Sigma B_\lambda$ with B_λ a closed subgroup of H_λ for each λ. Let the corresponding decomposition of G, given by Proposition* 6(iii), *be $G = \Pi G_\lambda$. Then $A = \Pi A_\lambda$, where $A_\lambda = A \cap G_\lambda$.*

Proof. We have $G_\lambda = (\sum_{\mu \neq \lambda} H_\mu)^0$, so $A_\lambda = A \cap G_\lambda$ is the annihilator of B_λ in G_λ. Hence ΠA_λ annihilates B, that is, $\Pi A_\lambda \subset A$. On the other hand, since A annihilates B_λ for each λ, we have $A \subset \Pi A_\lambda$. Hence $A = \Pi A_\lambda$ algebraically, and its topology, being induced from $G = \Pi G_\lambda$, is the product topology.

PROPOSITION 8. *Let $\phi: G \times H \to \mathbf{T}$ be a duality and suppose that G is a product of locally compact groups. Then, for every closed subgroup A of G, we have $A^{00} = A$.*

Proof. Clearly $A^{00} \supset A$, so we suppose that g is an element of G not in A and show that $g \notin A^{00}$. By Theorem 2 of (5), since G is a product of locally compact groups, there is a character χ of G which takes A but not g to zero. Since ϕ is a duality, χ is induced by an element h of H, which must be in the annihilator A^0 of A. But

$$\phi(g,h) = \chi(g) \neq 0,$$

so $g \notin A^{00}$.

4. *Closed subgroups and quotients of products.* We are now in a position to translate the results of section 2 into results about products.

THEOREM 2. *Let $G = \prod_{i=1}^{\infty} \mathbf{R}_i$ be the product of a countable number of copies $\mathbf{R}_i$ of $\mathbf{R}$, and let A be a closed subgroup of G. Then there is a decomposition $G = \prod_{i=1}^{\infty} G_i$ as a product of subgroups G_i each topologically isomorphic to $\mathbf{R}$ such that $A = \Pi A_i$, where $A_i = A \cap G_i$. Hence A is topologically isomorphic to a countable (possibly finite) product of copies of $\mathbf{R}$ and $\mathbf{Z}$, and G/A is topologically isomorphic to a countable product of copies of $\mathbf{R}$ and $\mathbf{T}$.*

Proof. Let $H = \sum_{i=1}^{\infty} \mathbf{R}_i = \mathbf{R}^\infty$. Then, by Kaplan's theorem, there is a duality

$$\phi: G \times H \to \mathbf{T}$$

given by $\phi(g,h) = \sum_{i=1}^{\infty} g_i h_i \pmod 1$. Let $B = A^0$. Then B is a closed subgroup of H and therefore, by Theorem 1, there is a decomposition $H = \sum_{i=1}^{\infty} H_i$, with $H_i \cong \mathbf{R}$, such that $B = \sum_{i=1}^{\infty} B_i$, where B_i is a closed subgroup of H_i for each i. By Proposition 8, $B^0 = A$, so we may apply Propositions 6 and 7 to obtain decompositions $G = \prod_{i=1}^{\infty} G_i$ and $A = \prod_{i=1}^{\infty} A_i$, such that $A_i = A \cap G_i$ and such that ϕ induces dualities $\phi_i: G_i \times H_i \to \mathbf{T}$. Since $H_i \cong \mathbf{R}$, it follows that $G_i \cong \mathbf{R}^\wedge \cong \mathbf{R}$ for all i. Also, the closed subgroup B_i of $H_i \cong \mathbf{R}$ is isomorphic to $\mathbf{R}, \mathbf{Z}$ or $\{0\}$ and H_i/B_i is isomorphic to $\mathbf{R}$, $\mathbf{T}$ or $\{0\}$. But A_i is the annihilator of B_i under the duality $\phi_i: G_i \times H_i \to \mathbf{T}$, so it follows from the duality theory of locally compact groups that $A_i \cong (H_i/B_i)^\wedge \cong \mathbf{R}, \mathbf{Z}$ or $\{0\}$ and that

$$G_i/A_i \cong B^\wedge \cong \mathbf{R}, \mathbf{T} \text{ or } \{0\}.$$

Hence $A = \Pi A_i$ and $G/A \cong \Pi(G_i/A_i)$ have the stated form.

COROLLARY 1. *If G is a countable product of copies of $\boldsymbol{R}$ and $\boldsymbol{Z}$, then any closed subgroup of G is also a countable product of copies of $\boldsymbol{R}$ and $\boldsymbol{Z}$.*

Proof. G can be embedded as a closed subgroup in a countable product of copies of $\boldsymbol{R}$.

COROLLARY 2. *If G is a countable product of copies of $\boldsymbol{R}$ and $\boldsymbol{T}$, then any Hausdorff quotient of G is a countable product of copies of $\boldsymbol{R}$ and $\boldsymbol{T}$.*

Proof. G is a quotient group of a countable product of copies of $\boldsymbol{R}$.

PROPOSITION 9. *Let $G = E \times C$ be a product of topological Abelian groups E and C, such that E is a product of copies of $\boldsymbol{R}$ and $\boldsymbol{Z}$, and C is a product of copies of $\boldsymbol{T}$. Let A be a closed subgroup of G. Then there is a decomposition $G = E' \times C$, where E' is a subgroup topologically isomorphic to E, such that $A = (A \cap E') \times (A \cap C)$.*

Proof. Let H be the character group of G, and let $\phi: G \times H \to \boldsymbol{T}$ be the natural pairing. By Kaplan's theorem, ϕ is a duality and H is a direct sum $H = V \oplus F$, where V is a sum of copies of $\boldsymbol{R}$ and $\boldsymbol{T}$, and F is a sum of copies of $\boldsymbol{Z}$, each with the asterisk topology. Clearly, V is divisible. Also, since the asterisk topology is at least as fine as the rectangular topology, F is discrete. We may therefore apply Proposition 5 to the closed subgroup $B = A^0$ of H, to obtain decompositions $H = V \oplus F'$ and $B = (B \cap V) \oplus (B \cap F')$, where $F' \cong F$. Again, by Proposition 8, $B^0 = A$, and the groups V and F' are reflexive (by Kaplan's theorem). Hence Proposition 6(ii) gives a decomposition $G = E' \times C$, where $E' = (F')^0 \oplus V^\wedge \cong E$, and Proposition 7 gives the decomposition $A = (A \cap E') \times (A \cap C)$.

THEOREM 3. *Let $G = E \times C$, where E is a product of copies of $\boldsymbol{R}$ and $\boldsymbol{Z}$, and C is a product of copies of $\boldsymbol{T}$. Then*

(i) *any connected closed subgroup A of G is of the form $A = A' \times C'$, where A' is a product of copies of $\boldsymbol{R}$, and $C' = A \cap C$ is a connected compact group;*

(ii) *if E is a countable product of copies of $\boldsymbol{R}$ and $\boldsymbol{Z}$, then any closed subgroup A of G is the product of the compact group $A \cap C$ and a countable product of copies of $\boldsymbol{R}$ and $\boldsymbol{Z}$.*

Proof. In both cases we have, by Proposition 9, decompositions $G = E' \times C$ and $A = A' \times C'$, where $C' = A \cap C$, $E' \cong E$ and A' is a closed subgroup of E'. If A is connected, so are A' and C', and the projections of A' onto the factors of E' of type $\boldsymbol{Z}$ are trivial. Hence A' is a connected subgroup of a product of copies of $\boldsymbol{R}$ and is therefore itself a product of copies of $\boldsymbol{R}$ by Theorem A of (3). In part (ii) of the theorem, A' is a closed subgroup of a countable product of copies of $\boldsymbol{R}$ and $\boldsymbol{Z}$, so is itself a countable product of copies of $\boldsymbol{R}$ and $\boldsymbol{Z}$ by Corollary 1 of Theorem 2.

We now introduce a category $\mathscr{D}_\Pi$, whose objects are all those topological Abelian groups which are products of compact groups and a countable (possibly finite) number of copies of $\boldsymbol{R}$ and $\boldsymbol{Z}$. The morphisms of $\mathscr{D}_\Pi$ are the continuous homomorphisms.

THEOREM B. *Closed subgroups and Hausdorff quotients of groups in $\mathscr{D}_\Pi$ are again in $\mathscr{D}_\Pi$.*

Proof. Every compact Abelian group can be embedded as a closed subgroup in a product of copies of $\boldsymbol{T}$ (by Pontrjagin duality, since every discrete Abelian group is a quotient of a sum of copies of $\boldsymbol{Z}$). Also any countable product of copies of $\boldsymbol{Z}$ can be embedded as a closed subgroup in a countable product of copies of $\boldsymbol{R}$. Hence any group G in $\mathscr{D}_\Pi$ is a closed subgroup of a group $G' = E \times C$, where E is a countable product of copies of $\boldsymbol{R}$, and C is a product of copies of $\boldsymbol{T}$. Any closed subgroup of G is a closed subgroup of G' and is therefore in $\mathscr{D}_\Pi$ by Theorem 3(ii). Any Hausdorff quotient G/H of G is topologically isomorphic to a closed subgroup of G'/H, so it is enough to show that G'/H is in $\mathscr{D}_\Pi$. By Proposition 9, we may assume that $H = (H \cap E) \times (H \cap C)$ and therefore that $G'/H \cong (E/(H \cap E)) \times (C/(H \cap C))$. But $C/(H \cap C)$ is compact and $E/(H \cap E)$ is a countable product of copies of $\boldsymbol{R}$ and $\boldsymbol{T}$ (Theorem 2), so the theorem follows, since a product of copies of $\boldsymbol{T}$ is compact.

5. *Strong duality.* It is well-known that for a locally compact Abelian group G, the natural duality $G \times G^\wedge \to \boldsymbol{T}$ induces dualities between appropriate subgroups and quotient groups of G and $G^\wedge$. A similar statement was proved by Varopoulos for $(\mathscr{L}_\infty)$-groups in (12) by measure-theoretical methods. In this final section, we shall prove the corresponding result for groups in the categories $\mathscr{D}_\Sigma$ and $\mathscr{D}_\Pi$ by an elementary argument, and hence show that these categories are dual in a strong sense.

Two important properties that the characters of a topological group D may or may not have, are the following:

X(1). For each closed subgroup C of D and each element d of D not in C, there is a character of D taking C, but not d, to zero.

X(2). Every character of every closed subgroup of D can be extended to a character of D.

PROPOSITION 10(i). *Properties* X(1) *and* X(2) *are each inherited by closed subgroups and Hausdorff quotients.*

(ii) *All groups in* $\mathscr{D}_\Sigma$ *or* $\mathscr{D}_\Pi$ *have properties* X(1) *and* X(2).

Proof (i). This is an easy consequence of the definitions.

(ii). Any group in $\mathscr{D}_\Pi$ is a product of locally compact groups, and properties X(1) and X(2) were proved for such products in ((5), Theorems 1 and 2). Any group in $\mathscr{D}_\Sigma$ is an $(\mathscr{L}_\infty)$-group in the sense of (12), and properties X(1) and X(2) were proved for $(\mathscr{L}_\infty)$-groups in ((12), Theorem p. 509). We give a simpler proof for $\mathscr{D}_\Sigma$-groups. Every group in $\mathscr{D}_\Sigma$ is a quotient group of a group of the type $\boldsymbol{R}^\infty \oplus F$, where F is a discrete free Abelian group. It is therefore enough, by (i), to prove that $H = \boldsymbol{R}^\infty \oplus F$ has properties X(1) and X(2). Let B be any closed subgroup of H. By Proposition 5 and Theorem 1, H has a decomposition $H = \sum_{i=0}^{\infty} H_i$, where $H_0 \cong F$ and $H_i \cong \boldsymbol{R}$ for $i \geqslant 1$, such that $B = \sum_{i=0}^{\infty} B_i$ with B_i a closed subgroup of H_i for each i. Now the groups H_i are all locally compact and so have properties X(1) and X(2) relative to the closed subgroups B_i. Since H is the coproduct of the H_i, and B is the coproduct of the B_i (Proposition 1, Corollary), it follows easily that H has properties X(1) and X(2) relative to B, which was an arbitrary closed subgroup.

PROPOSITION 11. *Let* $\phi: G \times H \to \boldsymbol{T}$ *be a duality and suppose that both* G *and* H *have properties* X(1) *and* X(2). *Then*

(i) $A^{00} = A$ *and* $B^{00} = B$ *for all closed subgroups* A *and* B *of* G *and* H, *respectively*;

(ii) *for any closed subgroups* A, B *of* G, H *respectively, with* $A^0 = B$ *and* $B^0 = A$, ϕ *induces open isomorphisms*

$$\phi^A: A \to (H/B)^\wedge, \quad \phi^B: B \to (G/A)^\wedge$$

and continuous isomorphisms

$$\phi_A: G/A \to B^\wedge, \qquad \phi_B: H/B \to A^\wedge.$$

Proof. (i) The argument has already been given in Proposition 8. It depends only on property X(1).

(ii) Since ϕ is a duality, it induces a topological isomorphism $\phi^G: G \to H^\wedge$. By restriction of characters to B, we obtain a continuous homomorphism $H^\wedge \to B^\wedge$ and, composing this with ϕ^G, we obtain a continuous homomorphism $G \to B^\wedge$ whose kernel is $B^0 = A$ and whose image is $B^\wedge$, by property X(2) for H. Hence the induced map $\phi_A: G/A \to B^\wedge$ is a continuous isomorphism and a similar argument applies to $\phi_B: H/B \to A^\wedge$. On the other hand, the topological isomorphism $\phi^G: G \to H^\wedge$ induces a topological isomorphism $A \to C$, where C is the subgroup of $H^\wedge$ consisting of all characters of H induced by elements of A. Now the quotient map $q: H \to H/B$ induces a continuous homomorphism $\theta: (H/B)^\wedge \to H^\wedge$, which is composition with q. Clearly θ is an injection, and its image is the group of all characters of H which vanish on B. These are precisely the characters induced by elements of $B^0 = A$. Thus θ induces a continuous isomorphism $(H/B)^\wedge \to C$, and it follows that $\phi^A: A \to (H/B)^\wedge$ is an open isomorphism. The same argument applies to ϕ^B.

We shall say that a duality $\phi: G \times H \to T$ is a *strong duality* if

(i) $A = A^{00}$ and $B^{00} = B$ for all closed subgroups A of G and B of H, and

(ii) for any closed subgroups A of G and B of H with $A^0 = B$, $B^0 = A$, the induced pairings $A \times (H/B) \to T$ and $(G/A) \times B \to T$ are dualities. This second condition is equivalent to the assertion that the maps ϕ^A, ϕ^B, ϕ_A and ϕ_B of Proposition 11 are topological isomorphisms.

PROPOSITION 12. *Let* $\phi: G \times H \to T$ *be a duality and suppose that* (i) *both* G *and* H *have properties* X(1) *and* X(2) *and* (ii) *every closed subgroup and every Hausdorff quotient of* G *and of* H *is reflexive. Then* ϕ *is a strong duality.*

Proof. Let A and B be closed subgroups of G and H respectively. By Proposition 11(i), we have $A^{00} = A$ and $B^{00} = B$. Taking $B = A^0$, and hence $A = B^0$, Proposition 11(ii) gives algebraic isomorphisms $\phi^A: A \to (H/B)^\wedge$ and $\phi_B: H/B \to A^\wedge$, of which ϕ^A is open and ϕ_B is continuous. Now the dual of ϕ_B is a continuous isomorphism

$$\phi_B^\wedge: A^{\wedge\wedge} \to (H/B)^\wedge,$$

and since by hypothesis A is reflexive, this induces a continuous isomorphism $A \to (H/B)^\wedge$, which clearly coincides with ϕ^A. Thus ϕ^A is both open and continuous, and so is a homeomorphism. Hence the dual of ϕ^A is a topological isomorphism $(H/B)^{\wedge\wedge} \to A^\wedge$, and since H/B is reflexive, this shows that $\phi^B: H/B \to A^\wedge$ is a topological isomorphism. The same arguments apply to ϕ^B and ϕ_A.

COROLLARY. *Every duality between a group in $\mathscr{D}_\Pi$ and a group in $\mathscr{D}_\Sigma$ is a strong duality.*

Proof. Let $\phi: G \times H \to T$ be a duality with $G \in \mathscr{D}_\Pi$ and $H \in \mathscr{D}_\Sigma$. Then G and H satisfy X(1) and X(2), by Proposition 10. Closed subgroups and Hausdorff quotients of G or H are again in $\mathscr{D}_\Pi$ or $\mathscr{D}_\Sigma$, respectively, by Theorems A and B, and are therefore reflexive, by Kaplan's theorem. We sum up these results in a categorical duality:

THEOREM C. *The contravariant functor D, taking each topological Abelian group to its dual, induces functors $D_1: \mathscr{D}_\Sigma \to \mathscr{D}_\Pi$ and $D_2: \mathscr{D}_\Pi \to \mathscr{D}_\Sigma$, such that $D_1 \circ D_2$ and $D_2 \circ D_1$ are naturally equivalent to identity functors. Moreover, D_1 and D_2 take closed inclusions to Hausdorff quotients and Hausdorff quotients to closed inclusions.*

Remark. L. J. Sulley has pointed out to us that since Banach spaces are reflexive (in the sense of character theory, see (14)), the example give by R. C. Hooper in (13), p. 254, of a Banach space C_0 not satisfying condition X(1) with respect to a closed subgroup K_1, shows that a Hausdorff quotient of a reflexive group need not be reflexive. Thus 'strong duality' is a strictly stronger property than 'duality', that is to say, duality is not in general inherited by closed subgroups and Hausdorff quotients. However, it is interesting to note that strong duality *is* so preserved. More precisely:

PROPOSITION 13. *Let $\phi: G \times H \to T$ be a strong duality between topological Abelian groups and let A, B be closed subgroups of G, H, with $B = A^0$ and $A = B^0$. Then the induced dualities $\psi: A \times (H/B) \to T$ and $\psi': (G/A) \times B \to T$ are strong.*

Proof. It is enough to show that ψ is strong. One shows easily that G and H have properties X(1) and X(2); these are inherited by A and H/B. If C is any closed subgroup of A, its annihilator under ψ is C^0/B and we have to show that the induced pairings $\sigma: C \times \{(H/B)/(C^0/B)\} \to T$ and $\tau: (A/C) \times (C^0/B) \to T$ are dualities. Since $(H/B)/(C^0/B) \cong H/C^0$, σ is essentially the pairing $C \times (H/C^0) \to T$ induced by ϕ and is a duality because ϕ is strong. As for τ, we may apply Proposition 11 to the pairing $\psi: A \times (H/B) \to T$ to show that the induced maps

$$\psi_C: A/C \to (C^0/B)^\wedge \quad \text{and} \quad \psi^{C^0/B}: C^0/B \to (A/C)^\wedge$$

are, respectively, a continuous isomorphism and an open isomorphism. On the other hand, A/C is a closed subgroup of G/C, and its annihilator under the induced duality $\theta: (G/C) \times C^0 \to T$ is B. Since G/C and C^0 also inherit the properties X(1) and X(2) from G and H, we may apply Proposition 11 again to show that $\theta^{A/C}: A/C \to (C^0/B)^\wedge$ is an open isomorphism and $\theta_B: C^0/B \to (A/C)^\wedge$ is a continuous isomorphism. It is clear that $\theta^{A/C} = \psi_C$ and $\theta_B = \psi^{C^0/B}$, so both are topological isomorphisms and τ is a strong duality.

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