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Local Compactness and Free Products of Topological Groups

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ABSTRACT—It is proved here that a free product (free abelian product) of an infinite family of non-totally disconnected topological groups is never locally compact.

1. Introduction

Definition. Let $\{G_i : i \in I\}$ be a family of topological groups. Then the topological group F is said to be a *free product* of $\{G_i : i \in I\}$, denoted by $\prod^*_{i \in I} G_i$, if it has the properties:

(*a*) for each $i \in I$, G_i is a subgroup of F;

(*b*) F is generated algebraically by $\bigcup_{i \in I} G_i$;

(*c*) if for each $i \in I$, γ_i, is a continuous homomorphism of G_i into a topological group H, then there exists a continuous homomorphism Γ of F into H such that $\Gamma = \gamma_i$ on G_i, for each i.

Of course a free product is simply the coproduct in the category of all topological groups. Thus, if it exists, it is unique up to (a unique) isomorphism. The *free abelian product* is similarly defined and is the coproduct in the category of all abelian topological groups.

Free products of topological groups were first studied by Graev (1950). He showed that the free product of any family of Hausdorff groups exists and is Hausdorff. (A much simpler—but fallacious—proof of this result appears in Morris (1971). A simple proof of a special case is given in Ordman (1974).

We summarize the other background results in the following

Theorem. Let $\{G_i : i \in I\}$ be a family of (not necessarily Hausdorff) topological groups.

(i) Morris (1971). $\prod^*_{i \in I} G_i$ exists and has as its underlying group structure the free algebraic product of G_i, $i \in I$.

(ii) Hulanicki (1967) and Morris (1971). If each G_i is maximally periodic, then $\prod^*_{i \in I} G_i$ is maximally almost periodic.

(iii) Morris (1971). If each G_i is connected, then $\prod^*_{i \in I} G_i$ is connected.

(iv) Ordman (1974). If $\prod^*_{i \in I} G_i$ is Hausdorff, then each G_i is a closed subgroup.

(v) Ordman (1974). If at least two G_i are not $\{e\}$, then $\prod^*_{i \in I} G_i$ is not compact.

(vi) Ordman (1974). If at least two G_i are not discrete, then $\prod^*_{i \in I} G_i$ is not both locally compact and locally invariant.

Note that in our main theorem and in part (vi) of the above theorem a non-triviality condition is assumed. Some such condition is necessary, as the free product of discrete groups is discrete.

2. Results

Lemma. Let $\{G_i : i \in I\}$ be an infinite family of non-totally disconnected locally compact groups. Let G be the algebraic restricted direct product of $\{G_i : i \in I\}$ with the finest group topology which will induce the given topology on each G_i. Then G is not locally compact.

Proof. Suppose G is locally compact. Let C_i be the component of the identity in G_i, for each $i \in I$.

Case 1. There exists an infinite subset J of I such that for each $j \in J$, C_j has a subgroup R_j isomorphic to the additive group of reals with its usual topology. Of course each R_j has a subgroup Z_j isomorphic to the discrete group of integers.

Let $\prod^D_{i \in I} G_i$ denote the restricted direct product of $\{G_i : i \in I\}$, with the usual topology. Then there is a natural continuous algebraic isomorphism f of G onto $\prod^D_{i \in I} G_i$, such that $f|G_i$ is an isomorphism of G_i onto $f(G_i)$, for each $i \in I$.

Let A be the subgroup of G generated algebraically by $\bigcup_{j \in J} R_j$. Since R_j is locally compact, $f(R_j)$ is a closed subgroup of $f(G_j) = G_j$.

Thus $f(A)=\prod^D_{j\varepsilon J} f(R_j)$ is a closed subgroup of $\prod^D_{i\varepsilon I} G_i$. Therefore, A is a closed subgroup of G and consequently is locally compact.

Similarly if we define B to be the subgroup of G generated algebraically by $\bigcup_{j\varepsilon J} Z_j$, then B is locally compact. Since B is algebraically isomorphic to $\prod^D_{j\varepsilon J} Z_j$, this implies by Dudley (1961), that B has the discrete topology.

Since each R_j, $j\varepsilon J$, is connected, A is connected. Thus A is a compactly generated locally compact abelian group which has B as a discrete subgroup. By Corollary 1 of Morris (1972), B is finitely generated. This contradicts the definition of B. Thus *Case 1 cannot occur.*

Case 2. There exists an infinite subset K of I, such that each C_k is compact, $k\varepsilon K$.

Let C be the subgroup of G generated algebraically by $\bigcup_{k\varepsilon K} C_k$. Since C_k is compact, $f(C_k)$ is a closed subgroup of $f(G_k)=G_k$. Thus $f(C)=\prod^D_{k\varepsilon K} f(C_k)$ is a closed subgroup of $\prod^D_{i\varepsilon I} G_i$. Therefore C is a closed subgroup of G and consequently is locally compact. Clearly C is also connected.

By §4.13 of Montgomery and Zippin (1955), C has a maximal compact subgroup M. Since C_k is a normal subgroup of C, MC_k is a subgroup of C, for each $k\varepsilon K$. Noting that MC_k is compact and contains M, we see that $MC_k=M$. Thus C_k is a subgroup of M, for each $k\varepsilon K$. The family $\{C_k : k\varepsilon K\}$ generates C algebraically and so $M=C$; that is, C is compact.

However, if C is compact then $f(C)=\prod^D_{k\varepsilon K} f(C_k)$ is compact. This contradicts Theorem 6.2 of Hewitt and Ross (1963) which clearly implies a restricted direct product of an infinite family of non-trivial groups is never compact. This contradiction shows that *Case 2 cannot occur.*

Now, by §4.13 of Montgomery and Zippin (1955) and our assumptions on the family $\{G_i : i\varepsilon I\}$, if G is locally compact then either Case 1 or Case 2 must occur. Hence G is not locally compact.

Theorem. Let $\{G_i : i\varepsilon I\}$ be an infinite family of non-totally disconnected topological groups. Then $\prod^*_{i\varepsilon I} G_i$ is not locally compact.

Proof. Suppose $\prod^*_{i\varepsilon I} G_i$ is locally compact. Then by part (iv) of our preliminary theorem, each G_i is locally compact. Let G be defined as in the Lemma. Then clearly there is a natural continuous open homomorphism of $\prod^*_{i\varepsilon I} G_i$ onto G. Thus G is locally compact. This contradicts the Lemma. Hence $\prod^*_{i\varepsilon I} G_i$ is not locally compact.

Similarly we obtain the following:

Theorem. Let $\{G_i : i\varepsilon I\}$ be an infinite family of abelian non-totally disconnected topological groups. Then the free abelian products of $\{G_i : i\varepsilon I\}$ is not locally compact.

We conclude by drawing the reader's attention to some work which complements the results obtained here: Morris, Ordman and Thompson (1973) and Morris (1975).

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