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FREE TOPOLOGICAL GROUPS WITH NO SMALL SUBGROUPS

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ABSTRACT. The first author has shown that a quotient group of a topological group with no small subgroups *can* have small subgroups, thus answering a question of Kaplansky in the negative. The argument relied on showing that a free *abelian* topological group on any metric space has no small subgroups. Consequently any *abelian* metric group is a quotient of a group with no small subgroups. However metric groups with small subgroups exist in profusion! It is shown here that a necessary and sufficient condition for a free (free abelian) topological group on a topological space X to have no small subgroups is that X admits a continuous metric. Hence any topological group which admits a continuous metric is a quotient group of a group with no small subgroups.

1. **Introduction.** It is well known that a quotient group of a Lie group is a Lie group, or equivalently [4], that a quotient group of a locally compact group with no small subgroups is a locally compact group with no small subgroups. Kaplansky [9] asks: if G is a topological group with no small subgroups and H is a closed normal subgroup of G, is G/H (necessarily) a group with no small subgroups? This question was answered in the negative by Morris [15]. There were three steps in the argument:

(1) To show that a free abelian topological group on any metric space has no small subgroups.

(2) To note that any abelian topological group is a quotient of its free abelian topological group.

(3) To show that there exist metric abelian groups with small subgroups.

As a consequence every abelian metric group is a quotient group of a group with no small subgroups.

We investigate the questions: Can "abelian" be removed? Can "metric" be weakened? We answer both questions in the affirmative. Conditions (2)

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and (3) are readily generalized. However the necessary generalization of (1) requires a new approach as the original proof of (1) relies heavily on the "abelian" assumption.

Our work yields:

Theorem. *A necessary and sufficient condition for the free (free abelian) topological group* $F(X)$ $(A(X))$ *on a topological space* X *to have no small subgroups is that* X *admits a continuous metric.*

Corollary. *Let* X *be locally compact and paracompact. Then* $F(X)$ $(A(X))$ *has no small subgroups if and only if* X *is metric.*

Corollary. *Any topological group which admits a continuous metric is a quotient of a topological group with no small subgroups.*

The second corollary shows that a large class of topological groups are quotients of topological groups with no small subgroups. We leave unanswered the question: Precisely what groups are quotients of groups with no small subgroups?

2. **Notation and preliminaries.** All topological spaces considered will be assumed to be completely regular and Hausdorff. We assume familiarity with the notions of free topological group and free abelian topological group due to Markov [12]. (See [6].) Given a topological space X, we denote the free topological group on X by $F(X)$.

We will denote the identity element of a group by e. The set of all words in $F(X)$ of length $\leq n$ with respect to X will be denoted by $F_n(X)$. We note that

$$F_1(X) = X \cup X^{-1} \quad \text{and} \quad F_n(X) = (X \cup X^{-1} \cup e)^n$$

for all $n > 1$. Thus if X is compact, then each $F_n(X)$ is compact.

We will need the following basic structure theorem of Graev [5]. (For an alternative proof of a more general result see [11].)

Theorem A. *Let* X *be a compact space. Then a subset* A *of* $F(X)$ *is closed if and only if* $A \cap F_n(X)$ *is compact for all* n.

As mentioned in the introduction, it is easily shown that every topological group G is a quotient group of its free topological group $F(G)$ [6, §8.23(b)].

Definition. A topological group is said to have no small subgroups if there exists a neighbourhood of e which contains no nontrivial subgroup. These groups will be referred to as NSS groups.

We will denote the set $\{1, 2, 3, \cdots\}$ by N. The closure of a set X in a topological space will be denoted by $\mathrm{cl}(X)$.

3. Results.

Lemma 1. *Any NSS group G admits a continuous metric.*

Proof. Since G is NSS, there exists an open symmetric neighbourhood V of e which contains no nontrivial subgroups. Define $V_n = \{g: g^n \in V\}$, for $n \in N$. Clearly each V_n is an open neighbourhood of e such that $\bigcap_n V_n = \{e\}$. We now define symmetric neighbourhoods U_n of e inductively such that (i) $U_n \subseteq V_n$ and (ii) $U_{n+1}^2 \subseteq U_n$, for $n \in N$. By Theorem 8.2 of [6], these sets U_n can be used to define a continuous pseudo-metric d on G. Since $\bigcap_n U_n = \{e\}$, d is a metric.

Remark. The converse of Lemma 1 is false. For example, let G be any topological group which admits a continuous metric. Define G_0 to be the product group $\Pi_{i \in I} G_i$ given the Tychonoff topology, where G_i is a topological group isomorphic to G, and I is countably infinite. Then G admits a continuous metric but it is not an NSS group [15]. However the converse *is* true for free topological groups.

Theorem 1. *The following conditions are equivalent:*

(i) $F(X)$ *is NSS.*

(ii) $F(X)$ *admits a continuous metric.*

(iii) X *admits a continuous metric.*

(iv) $F(X)$ *admits a two-sided invariant metric.*

Proof. (i) $\Rightarrow$ (ii) is given by Lemma 1. (ii) $\Rightarrow$ (iii) is trivial. (iii) $\Rightarrow$ (iv) follows from the proof of Theorem 1 of [5]. (iv) $\Rightarrow$ (iii) is trivial. It remains to show that (iii) $\Rightarrow$ (i).

It is sufficient to consider the case when X is a metric space. To see this, let d be a continuous metric on the completely regular space X, and denote the set X with the metric d by Y. Then the identity mapping $i: X \to Y$ is continuous and one-to-one, and therefore extends to a continuous monomorphism $I: F(X) \to F(Y)$. Thus if $F(Y)$ is NSS, so is $F(X)$.

Now let (X, d) be a metric space and $\beta(X)$ be its Stone-Čech compactification [10]. Then X is a dense subspace of $\beta(X)$. Let ϕ be the imbedding $\phi: X \to \beta(X)$. There exists a continuous homomorphism $\Phi: F(X) \to F(\beta(X))$ such that $\Phi|X = \phi$. Clearly Φ is an algebraic isomorphism of $F(X)$ onto $\Phi(F(X)) = G(X)$, say. Thus to prove $F(X)$ is NSS it suffices to show $G(X)$ (given the subspace topology in $F(\beta(X))$) is NSS.

Let $\{\rho_j: j \in J\}$ be a family of continuous pseudo-metrics on $\beta(X)$ whose restriction to X defines the subspace topology on X. Since X has a metric topology we may assume J is countable. (See [2, Chapter IX, §4, Theorem 1].) Relabelling, we may assume J is a subset of the positive integers. Then

$$\rho = \sum_{j \in J} \frac{1}{2^j} \frac{\rho_j}{1 + \rho_j}$$

is a continuous pseudo-metric on $\beta(X)$ whose restriction to X is a metric defining the same topology as d. A careful examination of the proof of Theorem 1 of [5] yields: (a) ρ can be extended to a continuous pseudo-metric ρ^1 on $F(\beta(X))$; (b) the restriction of ρ to X, call it s, can be extended to a metric s^1 on $G(X)$; ρ^1 is an extension of s^1. Define open sets $O_n = \{g: g \in F(\beta(X)), \rho^1(g, e) < 1/n\}$, for $n \in N$. Then $\bigcap_{n=1}^{\infty} O_n \cap G(X) = \{e\}$. Put

$$A_n^i = \{g^{n+i}: g \in F_i(\beta(X)) - O_n\}, \quad \text{for } n \in N,\ i \in N.$$

Then (i) each A_n^i is compact; (ii) $e \notin A_n^i$ for all $n \in N$ and $i \in N$; (iii) any word in A_n^i has length $\geq n + i$; (iv) for any $x \in G(X)$, $x \neq e$, there exists $n \in N$ and $i \in N$ such that $x^{n+i} \in A_n^i$. [Since if $x \in G(X)$ there exists i such that $x \in F_i(\beta(X))$. Since $\bigcap_{n=1}^{\infty} O_n \cap G(X) = \{e\}$, there exists n such that $x \notin O_n$, so for sufficiently large i and n, $x \in F_i(\beta(X)) - O_n$. Hence $x^{n+i} \in A_n^i$ for sufficiently large i and n.] Put $A = \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} A_n^i$. By condition (iii) we have that $A \cap F_n(\beta(X)) = \bigcup_{i+k \leq n} A_k^i \cap F_n(\beta(X))$, which is clearly compact. Now by Theorem A, A is closed in $F(\beta(X))$. Thus by condition (ii), $U = G(X) - A$ is an open neighbourhood of e in $G(X)$. Further by condition (iv), U contains no nontrivial subgroups. Hence $G(X)$ is NSS and the proof is complete.

Corollary 1. *Let X be a locally compact paracompact space. Then $F(X)$ is NSS if and only if X is metrizable.*

Proof. It is readily seen that any locally compact space which admits a continuous metric is locally metrizable. Since by Theorem 2-68 of [7], any locally metrizable paracompact space is metrizable, Theorem 1 shows that $F(X)$ is NSS if and only if X is metrizable.

Corollary 2. *Let G be any locally compact group. Then $F(G)$ is NSS if and only if G is metrizable.*

The following example shows that Corollary 2 cannot be extended to k_ω-groups. (A topological space X is said to be a k_ω-space if $X = \bigcup_{n=1}^{\infty} X_n$, where each X_n is compact and A is closed in X if and only if $A \cap X_n$ is compact for all n. As examples we have all compact spaces and all connected locally compact groups [13], [18].)

Example. Let T denote the circle group with the usual compact topology. Then the free product [17], [16] $T * T$ is a k_ω-space group which is NSS [16]. Thus $T * T$ admits a continuous metric (and hence $F(T * T)$ is NSS), however it is shown in [18] that $T * T$ is not metrizable.

Corollary 3. *If G is any topological group which admits a continuous metric then G is a quotient of an NSS group.*

Proof. By §2, G is a quotient of $F(G)$, which by Theorem 1 is NSS.

Remark. We now have that a large class of topological groups are quotients of NSS groups. This, however, leaves unanswered the question: Precisely what topological groups are quotients of NSS groups? The next corollary provides some information.

Corollary 4. *Let G be a topological group. Then the following conditions are equivalent.*

(i) *G is a quotient group of an NSS group.*

(ii) *G is a quotient group of a topological group which admits a continuous metric.*

(iii) *G is a quotient space of a topological space which admits a continuous metric.*

(iv) *G is a quotient of a free topological group which is NSS.*

(v) *G is a quotient of a topological group which admits a two-sided invariant metric.*

Proof. (i) $\Rightarrow$ (ii) by Lemma 1. (ii) $\Rightarrow$ (iii) is trivial. To see that (iii) $\Rightarrow$ (iv), let G be a quotient space of a topological space X which admits a continuous metric. By Lemma 1 of [1], G is a quotient group of $F(X)$. By Theorem 1, $F(X)$ is NSS. (iv) $\Rightarrow$ (v) by Theorem 1. Finally we need (v) $\Rightarrow$ (i). We note that (v) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i), as required.

Recently Joiner [8] has shown that if a topological space X contains a G_δ-point, then each of the topological spaces $F_n(X)$ ($\subseteq F(X)$) contains G_δ-points. Our next corollary shows that if X is compact then e cannot be a G_δ-point in $F_n(X)$ unless X is metrizable.

Corollary 5. *Let X be compact. Then the following conditions are equivalent.*

(i) *X is metrizable;*

(ii) *$F_n(X)$ is metrizable for all $n \geq 1$;*

(iii) *$e \in F_2(X)$ is a G_δ-set.*

Proof. To see that (i) $\Rightarrow$ (ii), we note that, by Theorem 1, if X is metrizable then $F(X)$ admits a continuous metric. Thus each $F_n(X)$ admits a continuous metric. Since $F_n(X)$ is compact it is metrizable for every n. (ii) $\Rightarrow$ (iii) is trivial. Now assume (iii) is true. Let $f: X \times X \to F_2(X)$ be given by $f(x, y) = xy^{-1}$. Then $f^{-1}(\{e\}) = \Delta = \{(x, x): x \in X\}$. Since f is continuous and e is a G_δ-set, Δ is a G_δ-set in $X \times X$. Thus by [3, Problem 4, p. 253], X is metrizable.

In conclusion we note that all the above results are true for free *abelian* topological groups. In particular the analogue of Theorem 1 can be proved either by our method or by the method indicated in [15]. We also note that the above results also hold for *Graev* free topological groups [5].

Added in proof. Further comments on this topic are made in a paper, by the second author, which is entitled *Remarks on free topological groups with no small subgroups* and will appear in the Journal of the Australian Mathematical Society.

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