REMARKS ON VARIETIES OF LOCALLY CONVEX LINEAR TOPOLOGICAL SPACES

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1. *Introduction*

The study of varieties of linear topological spaces was initiated in [4]. (Selected results from [4] were announced in [3].)

Before commenting on our results we briefly introduce our notation and terminology.

A *variety* is defined to be a non-empty class of real Hausdorff locally convex linear topological spaces (LCS's) closed under the operations of taking subspaces (not necessarily closed), separated quotients, arbitrary products and isomorphic images. As examples we have

(a) the class $\mathscr{S}$ of all Schwartz spaces [6]

(b) the class $\mathscr{N}$ of all nuclear spaces [19]

(c) the class of all LCS's having the weak topology.

For any class $\mathscr{C}$ of LCS's, the *variety generated by* $\mathscr{C}$, denoted by $\mathscr{V}(\mathscr{C})$, is the smallest variety containing $\mathscr{C}$.

Notation. Let $\mathscr{C}$ be any class of LCS's. Then (a) $S(\mathscr{C})$, (b) $Q(\mathscr{C})$, (c) $C(\mathscr{C})$ and (d) $P(\mathscr{C})$ denote respectively the class of all LCS's isomorphic to (a) subspaces of LCS's in $\mathscr{C}$, (b) quotient spaces of LCS's in $\mathscr{C}$, (c) Cartesian products of LCS's in $\mathscr{C}$ and (d) products of finite families of LCS's in $\mathscr{C}$.

It was shown by S. A. Saxon [18] that if $\mathscr{B}$ is the class of all infinite-dimensional Banach spaces, then $\bigcap_{B\in\mathscr{B}} \mathscr{V}(B) \supseteq \mathscr{N}$. The main purpose of our §2 is to find a reasonable upper bound for $\bigcap_{B\in\mathscr{B}} \mathscr{V}(B)$. We prove that $\bigcap_{B\in\mathscr{B}} \mathscr{V}(B) \subseteq \mathscr{S}$. This leaves open the intriguing question: precisely what is $\bigcap_{B\in\mathscr{B}} \mathscr{V}(B)$? More specifically, we prove that if B is any reflexive Banach space then $\mathscr{V}(B) \cap \mathscr{V}(c_0) \subseteq \mathscr{S}$, and that if $1 < p \neq q < \infty$ then $\mathscr{V}(1_p) \cap \mathscr{V}(1_q) \subseteq \mathscr{S}$.

In §3 we report on recent progress on the questions asked in [4].

We will need the following basic theorem of [4].

THEOREM A. *Let $\mathscr{C}$ be any class of LCS's. Then*

(i) $\mathscr{V}(E) = QSC(\mathscr{C}) = SCQP(\mathscr{C})$;

(ii) *if E is any normed vector space in $\mathscr{V}(\mathscr{C})$, then $E \in QSP(\mathscr{C})$.*

2. *Varieties generated by Banach spaces and the Schwartz variety, $\mathscr{S}$*

LEMMA 1. *Let $\{F_i : i \in I\}$ be a family of LCS's and E a (linear topological) subspace of the product $\prod_{i\in I} F_i$. If f is a continuous linear operation of E into a Banach space B then there exists a finite subset J of I, a closed linear subspace S of $\prod_{j\in J} F_j$, and continuous linear operators $f_1 : E \to S$ and $f_2 : S \to B$ such that the diagram below commutes.*

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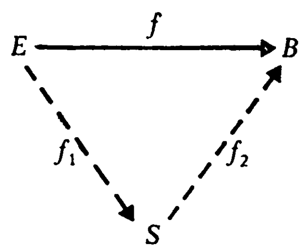


Proof. If U_B denotes the unit ball in B, then $f^{-1}(U_B)$ is a neighbourhood of zero in E. Since E is a subset of $\prod_{i\in I} F_i$, there exists a finite subset J of I such that $U \cap E \subseteq f^{-1}(U_B)$, where $U = \prod_{j\in J} U_j \times \prod_{j\in I-J} F_j$, each U_j being an open neighbourhood of zero in F_j.

Let $N = \{(x_j) \in E : x_j = 0 \text{ for all } j \in J\}$. Then N is a linear space with the property that $f(N) \subseteq U_B$. This implies that $f(N) = 0$. Thus for each $x = (x_i) \in E$, $f(x)$ is independent of the co-ordinates x_i for $i \in I - J$.

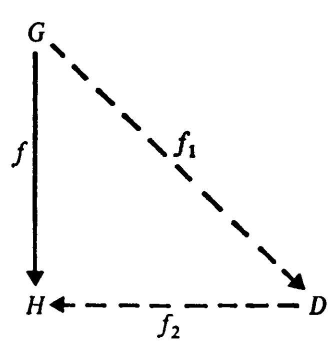
Let p be the canonical projection of $\prod_{i\in I} F_i$ onto $\prod_{j\in J} F_j$, f_1 the restriction of p to E, and $S_1 = f_1(E)$. Define a mapping $g : S_1 \to B$ by $g(s) = f(f_1^{-1}(s))$, for all $s \in S_1$. (We note that by our comments in the previous paragraph g is well-defined.) Clearly g is a continuous linear operator. Let S be the closure in $\prod_{j\in J} F_j$ of S_1. Then there exists a continuous linear operator $f_2 : S \to B$ such that the restriction to S_1 of f_2 is g. It is easily verified that the above diagram commutes, as required.

Remark. It is clear from the proof of the above lemma that the Banach space B can be replaced by any locally bounded complete Hausdorff linear topological space.

LEMMA 2. *Let B_1 and B_2 be Banach spaces. If for each Banach space $D \in \mathscr{V}(B_1)$ and each Banach space $E \in \mathscr{V}(B_2)$ every continuous linear operator $u : D \to E$ is compact, then $\mathscr{V}(B_1) \cap \mathscr{V}(B_2) \subseteq \mathscr{S}$.*

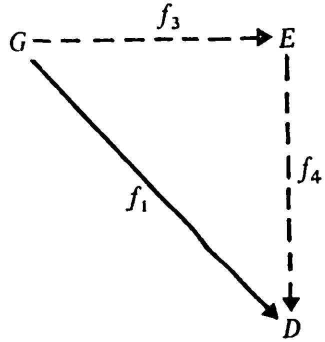
Proof. Let G be any LCS in $\mathscr{V}(B_1) \cap \mathscr{V}(B_2)$ and f a continuous linear operator of G into any Banach space H. To prove that G is a Schwartz space it suffices (p. 275 of [6]) to show that f is compact.

By Theorem A, $G \in SCQP(B_2)$. Noting that $QP(B_2)$ contains only Banach spaces, we see, by Lemma 1, that there exists a Banach space $D \in \mathscr{V}(B_2)$ and continuous linear operators $f_1 : G \to D$ and $f_2 : D \to H$ such that diagram (1) commutes.



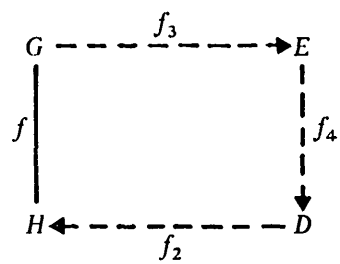
(1)

Applying Lemma 1 again, and using the fact that $G \in SCQP(B_1)$, we see that there exists a Banach space $E \in \mathscr{V}(B_1)$ and continuous linear operators $f_3 : G \to E$ and $f_4 : E \to D$ such that the diagram (2) commutes



(2)

Clearly then diagram (3) commutes



(3)

By our assumption, f_4 is compact. Consequently f is compact, as required.

THEOREM 1. *Let B be any Banach space. Then either* $c_0 \in \mathscr{V}(B)$ *or*

$$\mathscr{V}(B) \cap \mathscr{V}(c_0) \subseteq \mathscr{S}.$$

Proof. Suppose $\mathscr{V}(B) \cap \mathscr{V}(c_0)$ contains a non-Schwartz space. By Lemma 2, there exist Banach spaces D in $\mathscr{V}(c_0)$ and E in $\mathscr{V}(B)$, respectively, such that some continuous linear operator $u : D \to E$ is not compact.

Since $D \in \mathscr{V}(c_0)$, Theorem A implies that $D \in QSP(c_0) = QS(c_0)$, as c_0 is isomorphic to its own square. Thus there exists a closed subspace A of c_0 and a continuous linear operator f of A onto D. Noting that f is a quotient mapping, we see that the continuous linear operator $uf : A \to E$ is not compact. By [**16**; Remark 4, p. 212], this implies that E has a subspace isomorphic to c_0. Thus $c_0 \in \mathscr{V}(E) \subseteq \mathscr{V}(B)$, as required.

COROLLARY 1. *If B is any infinite-dimensional Banach space in* $\mathscr{V}(c_0)$ *then* $\mathscr{V}(B) = \mathscr{V}(c_0)$.

COROLLARY 2. *Let B be an infinite-dimensional Banach space which is either* (i) *reflexive,* (ii) *quasi-reflexive* [**1**] *or* (iii) *has separable second dual. Then* $\mathscr{V}(B) \cap \mathscr{V}(c_0) \subseteq \mathscr{S}$.

Proof. It is clear from Theorem A that if B has any one of the three properties named then so does every Banach space in $\mathscr{V}(B)$. Since c_0 does not possess any of these properties, $c_0 \notin \mathscr{V}(B)$. The result now follows from Theorem 1.

Remark. It was shown by Saxon [18] that if B is any infinite-dimensional Banach space, then $\mathscr{V}(B) \supset \mathscr{N}$, where $\mathscr{N}$ denotes the variety of all nuclear spaces. Thus if $\mathscr{B}$ is the class of all infinite-dimensional Banach spaces, then $\bigcap_{B \in \mathscr{B}} \mathscr{V}(B) \supseteq \mathscr{N}$.

In view of Corollary 2, we now have a nice upper bound for $\bigcap_{B \in \mathscr{B}} \mathscr{V}(B)$; namely, $\bigcap_{B \in \mathscr{B}} \mathscr{V}(B) \subseteq \mathscr{S}$.

Question 1. Is $\bigcap_{B \in \mathscr{B}} \mathscr{V}(B)$ equal to $\mathscr{N}$, $\mathscr{S}$ or neither?

Recently Randtke [15] has shown that $\mathscr{V}(c_0) \supseteq \mathscr{S}$. This partially answers Question 3 of [4]. We are left with

Question 2. For what Banach spaces B is it true that $\mathscr{V}(B) \supset \mathscr{S}$? In particular, if $1 < p < \infty$ does $\mathscr{V}(1_p) \supset \mathscr{S}$?

We now mention two (as yet unpublished) results of Professor H. P. Rosenthal.

(R_1) *If E is any infinite-dimensional Banach space in $QS(1_p)$, $1 < p < \infty$, then E has a subspace isomorphic to 1_p.*

(R_2) *If $1 < p < q < \infty$ and E and F are Banach spaces in $QS(1_p)$ and $QS(1_q)$ respectively, then every continuous linear operator $u : F \to E$ is compact.*

Using Theorem A, the fact that 1_p is isomorphic to its own square and (R_1) we have

THEOREM 3. *If $1 < p < \infty$ and B is any infinite-dimensional Banach space in $\mathscr{V}(1_p)$, then $\mathscr{V}(B) = \mathscr{V}(1_p)$.*

Using [4; Theorem 4.6] and Theorem 3 we obtain

COROLLARY 3. *If $1 < p \neq q < \infty$, then any Banach space in $\mathscr{V}(1_p) \cap \mathscr{V}(1_q)$ is finite-dimensional.*

Remark. Corollary 3 answers affirmatively Question 4 of [4]. A stronger result is presented in Theorem 4.

THEOREM 4. *If $1 < p \neq q < \infty$, then $\mathscr{V}(1_p) \cap \mathscr{V}(1_q) \subseteq \mathscr{S}$. (Cf. [2].)*

Proof. Without loss of generality, assume $p < q$. Suppose $\mathscr{V}(1_p) \cap \mathscr{V}(1_q)$ contains a non-Schwartz space. By Lemma 2, this implies that there exist Banach spaces D and E in $\mathscr{V}(1_q)$ and $\mathscr{V}(1_p)$, respectively, such that some continuous linear operator $u : D \to E$ is not compact. By Theorem A and the fact that 1_p and 1_q are isomorphic to their own squares we have that $D \in QS(1_q)$ and $E \in QS(1_p)$. However this contradicts (R_2). Hence $\mathscr{V}(1_p) \cap \mathscr{V}(1_q) \subseteq \mathscr{S}$.

In conclusion we note that Randtke [14] has found concrete examples of a universal generator for the variety of all Schwartz spaces. This answers Question 2 of [4]. (An existence proof was given in [4].)

3. *Miscellaneous results*

This section is concerned primarily with commenting on questions asked in [**4**]. We begin with an examination of the following remark which appeared at the end of §4 of [**4**]:

" If B is a quasi-reflexive Banach space then for each n, $B^{(2n)}$, the $(2n)$-th dual of B, is in $\mathscr{V}(B)$. How close does this phenomenon come to characterizing quasi-reflexivity? "

Notation: Whenever E is a Banach space we denote its Banach space dual by E'.

THEOREM 5. *Let E and F be Banach spaces. If $E \in \mathscr{V}(F)$, then $E' \in \mathscr{V}(F')$. Hence, if $\mathscr{V}(E) = \mathscr{V}(F)$, then $\mathscr{V}(E') = \mathscr{V}(F')$.*

Proof. If $E \in \mathscr{V}(F)$ then, by Theorem A, $E \in QSP(F)$. Thus, using the rules on duality for normed spaces, $E' \in SQP(F')$. Hence $E' \in \mathscr{V}(F')$. The final statement of the theorem is now obvious.

Example. Let $\mathscr{M}([0, 1])$ denote the space of Radon measures on $[0, 1]$. The Riesz-Markov Representation theorem says that $\mathscr{M}([0, 1])$ is the strong dual of $C([0, 1])$. It was noted in [**4**] that $\mathscr{V}(C([0, 1])) = \mathscr{V}(1_1)$. Therefore, by Theorem 5, $\mathscr{V}(\mathscr{M}([0, 1])) = \mathscr{V}(1'_1) = \mathscr{V}(1_\infty)$.

Remark. If B is any Banach space then B is isomorphic to a subsapce of B''. Therefore $\mathscr{V}(B) \subseteq \mathscr{V}(B'')$. If $B'' \in \mathscr{V}(B)$, then $\mathscr{V}(B) = \mathscr{V}(B'')$. By Theorem 5 this implies that $\mathscr{V}(B) = \mathscr{V}(B^{(2n)})$, for all $n \geqslant 0$.

Thus we see that if B is a Banach space with the property that $B'' \in \mathscr{V}(B)$, then $B^{(2n)} \in \mathscr{V}(B)$, for all $n \geqslant 0$. We now display a Banach space with this property which is not quasi-reflexive. (*Cf.* [**5**].)

Let I be a countably infinite set, and for each $i \in I$ let X_i be the quasi-reflexive space described by James [**7**]. Let X be the 1_2-sum of these spaces. It was noted in [**5**] that $X'' = \Pi(X) \oplus 1_2$, where Π is the canonical map of X into X'', and that X has a subspace isomorphic to 1_2. From these remarks we see that X is not quasi-reflexive but $\mathscr{V}(X) = \mathscr{V}(X'')$.

We now give an example which shows that the converse of Theorem 5 is false.

Example. Let $E = c_0(R)$ and $F = C([0, 1])$. Then we have:

(a) $E \notin \mathscr{V}(F)$, since F is separable but E is not. (See Corollary 4.16 of [**4**].)

(b) $F \notin \mathscr{V}(E)$, since E is almost reflexive while F is not. (See [**9**], [**13**] and [**4**; Theorem 4.9].)

(c) $\mathscr{V}(E') = \mathscr{V}(F')$, since by Theorem 5 and [**4**; Corollary 4.18], we have

$$\mathscr{V}(E') = \mathscr{V}(c_0(R)') = \mathscr{V}(1_1(R)) = \mathscr{V}(1_\infty) = \mathscr{V}(C(0, 1])') = \mathscr{V}(F').$$

Remark. The extension of Theorem 5 to Fréchet spaces E, F and their strong duals E'_β, F'_β is not possible without some extra conditions. For example, let $E = R$ and $F = R^\infty$, the product of countably many copies of R. Then we have

(a) $\mathscr{V}(E) = \mathscr{V}(F)$ but

(b) $\mathscr{V}(E'_\beta) \subsetneqq \mathscr{V}(F'_\beta)$, by [4; Theorem 3.6].

It was noted above that if E and F are Banach spaces such that $\mathscr{V}(E') = \mathscr{V}(F')$, then $\mathscr{V}(E)$ and $\mathscr{V}(F)$ need not be comparable. However we do have

THEOREM 6. *Let E be a separable Banach space. Then* $\mathscr{V}(c_0) \subseteq \mathscr{V}(E)$ *if and only if* $\mathscr{V}(1_1) \subseteq \mathscr{V}(E')$.

Proof. Using Theorem 5 we only have to show that $\mathscr{V}(1_1) \subseteq \mathscr{V}(E')$ implies $\mathscr{V}(c_0) \subseteq \mathscr{V}(E)$. Now by Theorem A, $\mathscr{V}(1_1) \subseteq \mathscr{V}(E')$ implies $1_1 \in QSP(E')$. Using Lemma 3.1 of [12] we have that $1_1 \in SP(E')$ or equivalently $1_1 \in S((PE)')$. Now by Theorem IV.3 of [8], $c_0 \in Q(PE)$, so that $c_0 \in QP(E)$. Hence $c_0 \in \mathscr{V}(E)$ and the proof is complete.

We now turn briefly to Question 5 of [4]: "If $1 < p \neq q < \infty$, for what r does $1_r \in \mathscr{V}(\{1_p, 1_q\})$?" Our next theorem gives a partial answer.

THEOREM 7. *Let* $1 < p \neq 2 < \infty$. *Then* $1_r \in \mathscr{V}(\{1_p, 1_2\})$ *if and only if r is between p and* 2.

Proof. Both 1_2 and 1_p belong to $\mathscr{V}(L_p(0, 1))$; so if $1_r \in \mathscr{V}(\{1_p, 1_q\})$, then $1_r \in \mathscr{V}(L_p(0, 1))$, which by Theorem 4.7 of [4] yields that r is between p and 2.

Conversely it is clear from [17] that if r is between p and 2, then 1_r is a quotient of a subspace of $1_p \oplus 1_2$, and hence $1_r \in \mathscr{V}(\{1_p, 1_2\})$.

This suggests the following question:

Question 3. If $1 < p < \infty$, does $L_p(0, 1) \in \mathscr{V}(\{1_p, 1_2\})$?

Question 12 of [4] was concerned with the possibility of there existing a separable reflexive Banach space E such that $\mathscr{V}(E)$ contains all separable reflexive Banach spaces. We give the following (negative) partial result in this direction.

THEOREM 8. *Let* $\mathscr{B}$ *be the class of all separable reflexive Banach space and let* $E \in \mathscr{B}$. *Suppose E is isomorphic to* E^2. *Then* $\mathscr{B} \nsubseteq \mathscr{V}(E)$.

Proof. If $\mathscr{V}(E)$ contains $\mathscr{B}$, then since E is isomorphic to E^2, $\mathscr{B} \subseteq QS(E)$. Now using the notation of [20] and [21], we have that if $\eta(E) = \alpha$ then for any $F \in QS(E)$, $\eta(F) \leqslant \eta(E)$, by [20; Proposition 2.3] and [21; Lemma 3]. Thus $\eta(F) \leqslant \alpha$ for each $F \in \mathscr{B}$, which contradicts Proposition 3.2 of [20], where it is exhibited that there exists, for each α, $F \in \mathscr{B}$ with $\eta(F) > \alpha$.

Remark. T. Figuel announced that there exists a reflexive Banach space which is not isomorphic to its own square. Thus it remains open whether or not the hypothesis that E be isomorphic to its own square can be removed from Theorem 8.

In conclusion we note that Question 11 of [4] which asked if distinct Orlicz sequence spaces are necessarily of incomparable linear dimension has recently been answered in the negative by [10] and [11]. So we record the question:

Question 4. If the reflexive Orlicz spaces 1^{Φ} and 1^{Ψ} are of incomparable linear dimension, does $\mathscr{V}(1^{\Phi}) \cap \mathscr{V}(1^{\Psi})$ consist entirely of Schwartz spaces?

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