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Extremally Disconnected Topological Groups

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Abstract.—It is shown that a metrizable extremally disconnected topological space is discrete. This general result is applied to show that a non-discrete finitely generated nilpotent topological group with a subgroup topology is not extremally disconnected.

1. Introduction

Arhangelskii [1] has shown that every compactly generated extremally disconnected topological group is finitely generated (a topological space is said to be extremally disconnected if the closure of every open set is open). It is not clear, however, whether there exists a non-discrete finitely generated extremally disconnected topological group.

We show that if an extremally disconnected topological space is metrizable then it is discrete (indeed it is sufficient that the space have a countable basis of open sets at each point); this result does appear to be known in the folklore but no elegant proof seems accessible. We apply this general result to the investigation of extremally disconnected topological groups in order to conclude that a finitely generated nilpotent topological group with a subgroup topology is not extremally disconnected, unless it is discrete.

A topological group with an open basis at the identity consisting of subgroups, that is, with a subgroup topology, clearly is "rather disconnected" in the sense that each set in the open basis is open and closed. It therefore seems reasonable to search for examples of extremally disconnected groups among the subgroup topologies. The negative conclusions reached tend to suggest that a subgroup topology on any group will not be extremally disconnected unless it is discrete.

2.

Theorem: If X is a metrizable extremally disconnected topological space then it is discrete.

Proof: Since X is metrizable, for each point $x \varepsilon X$ there is a sequence $W_1, W_2, \ldots$ of open neighbourhoods of x such that $\overline{W}_1, \overline{W}_2, \ldots$ is a base of neighbourhoods at x. As X is extremally disconnected each $\overline{W}_i$ is open, so $\overline{W}_1, \overline{W}_2, \ldots$ is an open basis at x.

Without loss of generality we can therefore assume that $W_1, W_2, \ldots$ are open and closed sets, that they are an open basis at x and that $X = W_0 \supseteq W_1 \supseteq W_2 \supseteq \ldots$ Put $T_i = W_{i-1} \setminus W_i$. So we have $W_{i-1} = W_i \cup T_i$, $W_i \cap T_i = \emptyset$ and T_i is open.

Let L be the even natural numbers, and M the odd natural numbers. Define $S = \bigcup_{l \varepsilon L} T_l$. We assert that (a) $\mathscr{C}S = \bigcup_{l \varepsilon M} T_l \cup \{x\}$ and (b) S is an open set whose closure is not open unless $\{x\}$ is open and X is thus discrete (where $\mathscr{C}S = X \setminus S$ is the complement of the set S).

To see (a) simply note that X is the disjoint union $\bigcup_{n=1}^{\infty} T_n \cup \{x\}$. To see (b) note that if $\mathscr{C}S$ contains a non-trivial open set containing $\{x\}$ then $\mathscr{C}S \supseteq W_i$ for some i. So $\mathscr{C}S$ contains T_l for all $l > i$ which clearly contradicts the definition of S. Hence the interior of $\mathscr{C}S$ is $\bigcup_{l \varepsilon M} T_l$; so $\overline{S} = \bigcup_{l \varepsilon L} T_l \cup \{x\}$ (where $\overline{S}$ is the closure of S). Similar reasoning shows that $\overline{S}$ contains no non-trivial open set containing $\{x\}$ so $\overline{S}$ is not open unless $\{x\}$ itself is open. Consequently, if X is extremally disconnected then $\{x\}$ must be open. Since this argument holds for all $x \varepsilon X$ we see that X is discrete.

Remark: In the theorem "metrizable" can be replaced by "first countable".

3.

We apply the general result to topological groups with a subgroup topology.

Proposition: Every Hausdorff subgroup topology on a finitely generated nilpotent group G is metrizable.

Proof: G is metrizable if there exists a countable open basis at the identity. This is certainly the case if G has only countably many subgroups but any subgroup of a finitely generated nilpotent group is finitely generated (see [2; at p. 182]) and there are only a countable number of finite subsets of G.

Corollary : No non-discrete Hausdorff subgroup topology on a finitely generated nilpotent group is extremally disconnected.

Remarks :

1. In the proposition and its corollary the words " finitely generated nilpotent " can be replaced by " supersoluble" (see [2 ; at p. 212]).
2. Noting that a quotient group of an extremally disconnected group with a subgroup topology is extremally disconnected our results extend to a somewhat larger class of groups than is explicitly mentioned.

4.

The technique employed in the proof of the theorem can be adapted to prove more general propositions. We illustrate such a generalization by means of an example which shows incidentally that metrizability is certainly not a necessary condition for the result of the theorem.

Example : Let $F_2(X)$ be the vector space over the field F_2, of two elements with basis an uncountable set X ; and let the subspaces $F_2(X\setminus Y)$ where Y is a *finite* subset of X be an open basis at 0 for the topology of the additive group $F_2(X)$. Then $F_2(X)$ is not metrizable. Let $x_1,x_2, \ldots$ be a sequence of distinct elements of X and write

$$W_0=F_2(X),\ W_1=F_2(X\setminus\{x_1\}),\ W_2=F_2(X\setminus\{x_1,x_2\}),\ \ldots$$

Then $K=\bigcap_{n=1}^{\infty} W_n=F_2(X\setminus\{x_1,x_2,\ldots\})$ is not an open set in $F_2(X)$.

As in the proof of the theorem we write $T_i=W_{i-1}\setminus W_i$ and consider the sets $S=\bigcup_{l\varepsilon L} T_l$ and $\mathscr{C}S=\bigcup_{l\varepsilon M} T_2 \cup K$. We find that if $F_2(X)$ is extremally disconnected then we obtain the contradiction that K contains an open subgroup and so is open.

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