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FREE TOPOLOGICAL GROUPS AND THE PROJECTIVE DIMENSION OF A LOCALLY COMPACT ABELIAN GROUP

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ABSTRACT. It is shown that a free topological group on a k_ω-space is a k_ω-space. Using this it is proved that if X is a k_ω-group then it is a quotient of a free topological group by a free topological group. A corollary to this is that the projective dimension of any k_ω-group, relative to the class of all continuous epimorphisms admitting sections, is either zero or one. In particular the projective dimension of a connected locally compact abelian group or a compact abelian group is exactly one.

1. **Introduction.** In [2] Graev showed that a free topological group on a compact Hausdorff space is a k_ω-space. It was observed [6] that Graev's proof worked not only for free topological groups but also for groups with the maximum topology relative to a given compact set, and several applications to free products of topological groups were obtained [8], [9], [11], [12].

It is shown here that any group having the maximum topology with respect to a k_ω-space generating it algebraically is itself a k_ω-space. In particular a free topological group on a k_ω-space is a k_ω-space. Thus our result generalizes Graev's result and our proof, which relies on known facts about k_ω-spaces, is easier than Graev's—or at least less delicate.

Recently Nummela [11] showed that the projective dimension of a compact abelian group, relative to the class of epimorphisms admitting sections, is exactly one. This was done by proving that a compact abelian group is a quotient group of a free abelian topological group by a free abelian topological group. The proof used Theorem 10 of [2] which provides a condition under which a subgroup of a free topological group on a compact set is a free topological group. (In general, such a subgroup is a free group but may have the wrong topology to be a free topological group.) We generalize Theorem 10 of [2] and thus conclude that the projective dimension of an abelian k_ω-group is zero or one. In particular, the projective dimension of a connected locally compact abelian group or a compact abelian group is exactly one.

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2. **Notation and preliminaries.** We assume familiarity with the notions of free topological groups due to Markov [**4**] and Graev [**2**]. Given a completely regular space X we denote by FGX (ZGX) the Graev free (free abelian) topological group on X.

Recall that a k_ω-space is a Hausdorff topological space with compact subsets X_n such that (i) $X=\bigcup_{n=1}^{\infty} X_n$; (ii) $X_{n+1}\supseteq X_n$ for all n; (iii) a subset A of X is closed if and only if $A \cap X_n$ is compact for all n. (For information on k_ω-spaces see [**5**] and [**13**].) Whenever we say that a k_ω-space has decomposition $X=\bigcup X_n$, we mean that the X_n have properties (i), (ii) and (iii) above.

We will use the following properties of k_ω-spaces: the direct product of two k_ω-spaces is a k_ω-space; a quotient space of a k_ω-space is a k_ω-space; a closed subspace of a k_ω-space is a k_ω-space; if a function f defined on a k_ω-space $X=\bigcup X_n$ is continuous on each X_n, then it is continuous on X; if $X=\bigcup X_n$ is a k_ω-space then any compact subset Y of X is contained in some X_n; if $X=\bigcup X_n$ is a k_ω-space and $Y_1, Y_2, \cdots$ is an increasing sequence of compact subsets of X such that each X_n is contained in some Y_m then X also has decomposition $X=\bigcup Y_n$. By way of examples, we note that every connected locally compact group and every compact space is a k_ω-space.

In [**11**] Nummela introduced a concept of projective dimension for abelian topological groups. For our purposes it suffices to note that an abelian topological group is of projective dimension one if and only if it is not projective and it is a quotient group of a free abelian topological group by a free abelian topological group.

If G is a group and X is a subset of G, we denote by $G_n(X)$ the set of words in G of length not exceeding n with respect to X. We note that if G is a Markov or Graev free topological group on the Hausdorff space X then $G_n(X)$ is closed in G for all n. (This fact is stated in [**1**] and can be proved in a similar manner to Theorem 2.2 of [**6**].)

We will denote the identity of any group by e.

If X is a topological group, FGX is the Graev free topological group on X and x and y are in X, then we denote by $x \cdot y$ the product in FGX of x and y. We denote by xy the product of x and y in X.

3. **Results.**

THEOREM 1. *Let X be a k_ω-space and G a Hausdorff group which is generated algebraically by X and is such that the topology of G is the finest group topology which induces the same topology on X. Then G is a k_ω-space and a subset A of G is closed if and only if $A \cap G_n(X)$ is closed in $G_n(X)$, for all n.*

PROOF. Let $X=\bigcup X_n$ be a decomposition of the k_ω-space X into compact sets, so that $G=\bigcup X^n$, where the sets $X^n=G_n(X_n)$ are compact. We must show this is a decomposition of G as a k_ω-space.

Let τ be the given topology on G and define a topology τ' on G as follows: A is a closed set in τ' if and only if $A\cap X^n$ is compact, for all n. Clearly τ' is a Hausdorff topology and $\tau'\supseteq\tau$. Indeed (G, τ') is a k_ω-space. We wish to show that (G, τ') is a topological group. To do this we must show that the map $f:(G, \tau')\times(G, \tau')\to(G, \tau')$ given by $f(x, y)=xy^{-1}$ is continuous.

Since (G, τ') is a k_ω-space, $(G, \tau')\times(G, \tau')$ is also a k_ω-space. Therefore, to show that f is continuous we only have to show that f is continuous on all compact subsets of $(G, \tau')\times(G, \tau')$.

Let K be a compact subset of $(G, \tau')\times(G, \tau')$. Then $K\subseteq K_1\times K_1$, where K_1 is a compact subset of (G, τ'). Since (G, τ') is a k_ω-space with decomposition $G=\bigcup X^n$, we see that $K_1\subseteq X^n$, for some n. Thus $f(K)\subseteq f(K_1\times K_1)\subseteq f(X^n\times X^n)\subseteq X^{2n}$. Noting that K is compact and $\tau'\supseteq\tau$, we see that K has the same induced topology as a subset of $(G, \tau')\times(G, \tau')$ as it has as a subset of $(G, \tau)\times(G, \tau)$. Similarly X^{2n} has the same induced topology as a subset of (G, τ') as it has a subset of (G, τ). Thus, since (G, τ) is a topological group, $f: K\to X^{2n}$ is continuous. So f is continuous on all compact subsets of $(G, \tau')\times(G, \tau')$. Hence (G, τ') is a topological group.

Now a subset A of X is closed in the topology induced on X from (G, τ') if and only if $A=A'\cap X$ where $A'\cap X^n$ is compact for every n. Since $X_n\subset X^n$, $A'\cap X^n$ compact implies $A'\cap X_n$ is compact for every n. But $X_n\subset X$, so $A'\cap X_n=A'\cap X\cap X_n=A\cap X_n$ is compact, for each n. Now since $X\subset(G, \tau)$ is a k_ω-space, A is closed in the topology induced on X from (G, τ). Noting that $\tau'\supseteq\tau$ we then see that (G, τ) and (G, τ') induce the same topology on X. However, by hypothesis, τ is the finest group topology on G inducing the given topology on X and $\tau'\supseteq\tau$. Hence $\tau'=\tau$.

Thus G is a k_ω-space and A is closed in G if and only if $A\cap X^n$ is compact, for all n.

Now let A be a subset of G such that $A\cap G_n(X)$ is closed in $G_n(X)$, for all n. Noting that $X^n\subseteq G_n(X)$, we have that $A\cap X^n$ is compact, for all n. Thus A is closed in G. The proof is complete.

REMARK. We note that the sets $G_n(X)$ (or even X itself) need not be closed subsets of G. For example, if G is the additive group of reals and $X=(-1, 1)$, then $G_n(X)=(-n, n)$ which is not closed even though Theorem 1 clearly applies.

COROLLARY 1. *Let X be a k_ω-space and G one of the following*: (i) *a Markov free topological group on X*; (ii) *a Markov free abelian topological* *group on X; (iii) a Graev free topological group on X; (iv) a Graev free abelian topological group on X. Then G is a k_ω-space and a subset A of G is closed if and only if $A \cap G_n(X)$ is closed in $G_n(X)$, for all n.*

THEOREM 2. *Let $X = \bigcup X_n$ be a decomposition of a k_ω-space X into compact sets. Let G be a Hausdorff group generated algebraically by X and let $X^n = G_n(X_n)$. If G has the property that a subset A of G is closed in G if and only if $A \cap X^n$ is compact for all n, then the topology of G is the finest group topology which induces the given topology on X.*

PROOF. Let τ be the given topology on G and $\tau' \supseteq \tau$ the finest group topology inducing the given topology on X. By the proof of Theorem 1, $A \subset G$ is closed in (G, τ') if and only if each $A \cap X^n$ is compact. But τ and τ' induce the same topology on X, hence on X_n and hence also on X^n. Thus $\tau' = \tau$ as desired.

THEOREM 3. *Let $X = \bigcup X_n$ be a k_ω-space. Let $Y \subset FGX$ be a subset such that $Y - \{e\}$ freely generates $G(Y)$, the subgroup of FGX generated by Y. Suppose $Y_1, Y_2, \cdots$ is a sequence of compact subsets of Y such that $Y = \bigcup Y_n$ is a k_ω-decomposition of Y inducing the same topology on Y that Y inherits as a subset of FGX. Put $X^n = G_n(X_n)$ and $Y^n = G_n(Y_n)$. If for each natural number n there is an m such that $G(Y) \cap X^n \subseteq Y^m$, then $G(Y)$ is the Graev free topological group on Y and both $G(Y)$ and Y are closed subsets of FGX.*

PROOF. It follows from the proof of Theorem 1 that, to prove $G(Y)$ is closed in FGX, we only have to show that $G(Y) \cap X^n$ is compact for each n. Now $G(Y) \cap X^n = G(Y) \cap X^n \cap Y^m = Y^m \cap X^n$, and hence is compact. Thus $G(Y)$ is closed in FGX. Similarly Y is closed in FGX.

Using Theorem 2, to prove $G(Y)$ is the Graev free topological group on Y, it suffices to show that a subset A of $G(Y)$ is closed if $A \cap Y^n$ is compact for all n. Consider $A \cap X^n$, for any n. There exists an m such that $G(Y) \cap X^n \subseteq Y^m$ and hence $A \cap X^n = A \cap X^n \cap G(Y) = A \cap X^n \cap Y^m = (A \cap Y^m) \cap X^n$. Since both $A \cap Y^m$ and X^n are compact, $A \cap X^n$ is compact, for all n. Thus A is a closed subset of FGX and the proof is complete.

COROLLARY 2 [**2**, THEOREM 10]. *Let X be a compact Hausdorff space and Y a compact subset of FGX which freely generates $G(Y)$. If for each n there is an m such that $G(Y) \cap G_n(X) \subset G_m(Y)$, then $G(Y)$ is the Graev free topological group on Y and it is closed in FGX.*

PROOF. Apply Theorem 3 with $X_n = X$ and $Y_n = Y$ for all n. We now generalize Proposition 1.6 of [**11**].

THEOREM 4. *Let the Hausdorff group X be a k_ω-space. Let $\psi: FGX \to X$ be the canonical quotient morphism. Then the kernel K of ψ is a Graev free topological group.*

PROOF. Since X is both a topological group and a k_ω-space there exists a sequence $\{X_n\}$ of compact sets such that (i) $X = \bigcup_{n=1}^{\infty} X_n$; (ii) $X_n \subseteq X_{n+1}$, for all n; (iii) if $x \in X_n$ and $y \in X_m$ then $xy \in X_{n+m}$; (iv) $x \in X_n$ implies the inverse (in X) of x is in X_n; (v) a subset A of X is closed if and only if $A \cap X_n$ is compact, for all n.

Define a map $\phi: X \times X \to FGX$ by $\phi(x,y) = x \cdot y \cdot (xy)^{-1}$ where $x \in X$, $y \in X$, $xy \in X$ and $(xy)^{-1}$ is the inverse in FGX of xy. Define $B = \phi(X \times X)$ and $B_n = \phi(X_n \times X_n)$. Then each B_n is compact and $B = \bigcup_{n=1}^{\infty} B_n$. It follows from Hall ([3, pp. 94–98]; see also [9, 1.3–1.6]) that K is freely generated by $B - \{e\}$. Put $B^n = G_n(B_n)$ and $X^n = G_n(X_n)$. We now prove (*).

For each n there is an m such that

$$(*) \qquad K \cap X^n \subseteq B^m.$$

Let $k \in K$, so $k = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, where $x_i \in X_n$ and $\varepsilon_i = \pm 1$, $i = 1, \cdots, n$. (Note x_i^{-1} denotes the inverse in FGX of x_i.) It is readily verified that $k = b_1 \cdots b_n$, where

$$b_j = (x_1^{\varepsilon_1} \cdots x_{j-1}^{\varepsilon_{j-1}}) \cdot (x_j)^{\varepsilon_j} \cdot (x_1^{\varepsilon_1} \cdots x_j^{\varepsilon_j})^{-1}, \qquad j > 1$$

and $b_1 = x_1^{\varepsilon_1} \cdot (x_1^{\varepsilon_1})^{-1}$. Here all operations inside parentheses are in X and all operations outside parentheses (inverting the third term and inverting x_j in the second term if $\varepsilon_j = -1$) are in FGX. [Recall that $(x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n})$ is the identity element.] We now observe that if $\varepsilon_j = +1$, $b_j = \phi(x_1^{\varepsilon_1} \cdots x_{j-1}^{\varepsilon_{j-1}}, x_j)$, while if $\varepsilon_j = -1$, $b_j = (\phi(x_1^{\varepsilon_1} \cdots x_j^{\varepsilon_j}, x_j))^{-1}$ the inverse being taken in FGX. We now note using the properties (iii) and (iv) of the sets X_n that, since each $x_i \in X_n$, each b_j is a word of three letters each of which is in (at worst) X_{n^2} and thus $b_j \in B_{n^2}$. But then $k \in G_n(B_{n^2}) \subset B^{n^2}$, as desired, proving (*).

Our next step is to show that B is closed in FGX. Now $B \cap X^n \subseteq B \cap (K \cap X^n) \subseteq B \cap B^m = B_m$. Thus $B \cap X^n = B_m \cap X^n$ which is clearly compact. Hence B is closed in FGX. Therefore B is a k_ω-space with decomposition $B = \bigcup (B \cap X^n)$. Now for each n, $B \cap X^n \subset B_m$, for some m; conversely each B_m, being compact, is contained in some $B \cap X^k$. Thus the decomposition $B = \bigcup B_n$ induces the same topology on B as $B = \bigcup (B \cap X^n)$, which induces the subspace topology on $B \subset FGX$.

Since we have now checked each hypothesis of Theorem 3, K is the Graev free topological group on B.

COROLLARY 3. *Let the Hausdorff abelian group X be a k_ω-space Let ZGX be the Graev free abelian topological group on X and $\Psi: ZGX \to X$ the* *canonical quotient morphism. Then the kernel K of Ψ is a Graev free abelian topological group.*

REMARK. We note that by Remark 2 of [7], Theorem 3 remains true if "Graev" is replaced throughout by "Markov".

COROLLARY 4. *Let the Hausdorff abelian group X be a k_ω-space. Then the projective dimension of X is either zero or one. In particular, the projective dimension of a connected locally compact abelian group or a compact abelian group is one.*

PROOF. This result follows immediately from Corollary 3 and the fact (Proposition 2.1 of [**11**]) that no locally compact abelian group is projective unless it is a discrete free abelian group.

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