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Invariant metrics on free topological groups

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For a completely regular space X , $G(X)$ denotes the free topological group on X in the sense of Graev. Graev proves the existence of $G(X)$ by showing that every pseudo-metric on X can be extended to a two-sided invariant pseudo-metric on the abstract group $G(X)$. It is natural to ask if the topology given by these two-sided invariant pseudo-metrics on $G(X)$ is precisely the free topological group topology on $G(X)$. If X has the discrete topology the answer is clearly in the affirmative. It is shown here that if X is not totally disconnected then the answer is always in the negative.

Introduction

In [8] and [2] Markov and Graev introduced their respective concepts of free topological group. Graev's concept is more general in the sense that every Markov free topological group is a Graev free topological group. Graev showed that for every completely regular space X his free topological group $G(X)$ always exists. His method of proof was to show that every pseudo-metric on X can be extended to a two-sided invariant pseudo-metric on the abstract group $G(X)$. These pseudo-metrics give rise to a locally invariant topology on the abstract group $G(X)$. It is natural to ask if $G(X)$ with its free topological group topology is locally invariant? This question is of interest since Joiner [4] and Abels [1] have recently investigated the topology on $G(X)$ using pseudo-metrics and related concepts.

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We show that if X is not totally disconnected then $G(X)$ is not locally invariant. (We note that some restriction on X is necessary since if X is discrete then $G(X)$ is discrete and hence locally invariant.) This result is also of interest as $G(X)$ is always maximally almost periodic [5] and the conditions "maximally almost periodic" and "locally invariant" are "close". For example, for connected locally compact groups the conditions coincide [3].

Notation and preliminaries

All topological spaces considered will be assumed to be completely regular and Hausdorff. We assume familiarity with the notions of free topological group due to Markov [8] and Graev [2]. Given a topological space X, we denote the Graev (Markov) free topological group on X by $G(X)$ $(F(X))$.

We will denote the identity element of a group by e. The set of all words in $G(X)$ of length $\leq n$ with respect to X will be denoted by $G_n(X)$. We note that $G_n(X) = (X \cup X^{-1})^n$, for each n. Thus if X is compact then each $G_n(X)$ is compact.

We will need the following basic structure theorem of Graev [2]. (For an alternative proof of a more general result, see [7].)

THEOREM A. *Let X be a compact space. Then a subset A of $G(X)$ is closed if and only if $A \cap G_n(X)$ is compact for all n.*

DEFINITION. A topological group G is said to be *locally invariant* if every neighbourhood of e contains an *invariant* neighbourhood of e; that is, a neighbourhood of e invariant under all inner automorphisms of G.

Note that a topological group is locally invariant if and only if its topology is given by a family of two-sided invariant pseudo-metrics.

We will use $[0, 1]$ to denote the unit interval of the reals with its usual compact topology. The closure of a set X in a topological space will be denoted by $\text{cl}.X$.

Results

LEMMA. *Let X be any non-totally disconnected space. Then there exists $x \in X$ and closed subsets A_n, $n = 1, 2, \ldots$, of X such that*

(i) $x \notin A_n$, for all n,

(ii) for any neighbourhood V of x there exists n such that $V \cap A_n \neq \emptyset$.

Proof. Since X is not totally-disconnected there exists an infinite closed connected subset Y of X.

Suppose that for each $y \in Y$ and every continuous function $f : Y \to [0, 1]$, there exists a neighbourhood V of y such that $f(V) = f(y)$. Since Y is completely regular there exists a function $f : Y \to [0, 1]$ such that $f(Y)$ contains at least two points. Clearly for any $p \in f(Y)$, $f^{-1}(p)$ is an open and closed proper subset of Y. This is a contradiction.

Thus there exists $y \in Y$ and a continuous function $f : Y \to [0, 1]$ such that for any neighbourhood V of Y, $f(V) \neq f(y)$. Let S_n be a countable base of open neighbourhoods at $f(y)$. Put $U_n = f^{-1}(S_n)$ and $A_n = Y - U_n$.

We note that each A_n is a closed subset of X, and $y \notin A_n$ for any n. If U is any neighbourhood in X of y, then $U \cap Y$ is a neighbourhood in Y of y. Thus $f(U \cap Y) \neq f(y)$. So there exists $v \in U \cap Y$ such that $f(v) \notin S_n$ for some n. Hence $(U \cap Y) \cap A_n \neq \emptyset$. Thus $U \cap A_n \neq \emptyset$ for some n.

THEOREM. *If X is a non-totally disconnected space then $G(X)$ is not locally invariant.*

Proof. By the lemma, there exists $x \in X$ and closed subsets A_n, $n = 1, 2, \ldots$, of X such that

(i) $x \notin A_n$ for any n and

(ii) for any neighbourhood V of x there exists an n such that $V \cap A_n \neq \emptyset$.

Now consider $G(X)$ where the point x is chosen to be the identity e . Let $\beta(X)$ be the Stone-Čech compactification of X [6]. The embedding $\phi : X \to \beta(X)$ can be extended to a continuous one-to-one homomorphism $\Phi : G(X) \to G(\beta(X))$. Put $B_n = \mathrm{cl.}(\Phi(A_n))$. Noting that e is not an isolated point of X and $e \notin A_n$, for any n , we can choose distinct points $x_n \in X\text{-}A_n$ such that $x_n \neq e$ for $n = 1, 2, \ldots$. Let $y_n = \Phi(x_n)$. Clearly the y_n are distinct points, $y_n \neq e$ and $y_n \in \beta(X)\text{-}B_n$, for $n = 1, 2, \ldots$.

Put $Y_n = y_1^{-1} \ldots y_n^{-1} B_n y_n \ldots y_1$. Then

(a) each Y_n is compact, since B_n is compact,

(b) every word in Y_n is of length $2n + 1$, and

(c) $\Phi^{-1}(Y_n) = x_1^{-1} \ldots x_n^{-1} A_n x_n \ldots x_1 = X_n$.

Let $B = \bigcup_{n=1}^{\infty} Y_n$ and $A = \bigcup_{n=1}^{\infty} X_n$. By condition (b), for each k , $B \cap G_k(\beta(X)) = \bigcup_{n=1}^{k} Y_n \cap G_k(\beta(X))$. Since both Y_n and $G_k(\beta(X))$ are compact, $B \cap G_k(\beta(X))$ is compact for each k . Thus by the Structure Theorem A, B is closed in $G(\beta(X))$. Noting that $A = \Phi^{-1}(B)$ we have A is closed in $G(X)$.

Define $U = G(X) - A$. Then U is an open neighbourhood of e . We show that U contains no invariant neighbourhoods of e . Suppose $U \supseteq V$, where V is an invariant neighbourhood of e . By condition (ii), $V \cap X \cap A_n \neq \emptyset$ for some n . Thus there exists $v \in V \cap X \cap A_n$. Therefore $x_1^{-1} \ldots x_n^{-1} v x_n \ldots x_1 \in X_n \subseteq A$. Hence $x_1^{-1} \ldots x_n^{-1} V x_n \ldots x_1 \nsubseteq V$. So V is not invariant.

COROLLARY. *If X is not totally disconnected then $F(X)$ is not locally invariant.*

We conclude our results with an application to the theory of k_ω-space groups.

DEFINITION. A topological space X is said to be a k_ω-space if $X = \bigcup_{n=1}^{\infty} X_n$ where each X_n is a compact and a subset A of X is closed if and only if $A \cap X_n$ is compact for all n.

As a prime example of k_ω-spaces we have any compact space and any connected locally compact group. For further information see [7], [9], and [10].

As mentioned in the introduction for connected locally compact groups the concepts "maximally almost periodic" and "locally invariant" coincide. One might also expect this to be the case for topological groups which are k_ω-spaces. Our final example shows that this is not true.

EXAMPLE. Let X be any compact space which is not totally disconnected. Then by the Structure Theorem A, $G(X)$ is a k_ω-space. (This is indeed the case even if X is only a k_ω-space which is not totally disconnected [7].) By [5], $G(X)$ is maximally almost periodic, but by our results it is not locally invariant.

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