

# Locally compact groups and $\beta$ -varieties of topological groups

by

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**Abstract.** It has previously been shown by the author that any abelian variety containing a connected compact group is a  $\beta$ -variety. As a consequence, it is seen that the varieties  $V(\mathbb{R})$  and  $V(T)$ , respectively generated by the additive group of reals and the circle group, are  $\beta$ -varieties. It is proved here that any variety generated by arcwise connected Hausdorff groups is a  $\beta$ -variety. From this it is deduced that connected locally compact groups generate  $\beta$ -varieties. It is also noted that any variety containing a non-abelian connected compact group is a  $\beta$ -variety.

**§ 1. Preliminaries.** A non-empty class  $V$  of topological groups (not necessarily Hausdorff) is said to be a *variety* if it is closed under the operations of taking subgroups, quotients, arbitrary cartesian products and isomorphic images. (See [8]–[11].)

We note that a variety (of topological groups) determines a variety of groups [15]; the latter is simply the class of groups which with some topology appear in the former.

The smallest variety containing a class  $\Omega$  of topological groups is said to be the *variety generated by*  $\Omega$  and is denoted by  $V(\Omega)$ . (See [2], [3] and [12]–[14].)

A *full variety* is a variety  $V$  which contains every topological group algebraically isomorphic to a member of  $V$ .

If  $V$  is a variety,  $X$  is a topological space and  $F$  is a member of  $V$ , then  $F$  is said to be a *free topological group of*  $V$  *on*  $X$ , denoted by  $F(X, V)$ , if it has the properties:

(a)  $X$  is a subspace of  $F$ .

(b)  $X$  generates  $F$  algebraically.

(c) For any continuous mapping  $\gamma$  of  $X$  into any group  $H$  in  $V$ , there exists a continuous homomorphism  $\Gamma$  of  $F$  into  $H$  such that  $\Gamma|_X = \gamma$ .

The following results on free topological groups are proved in [8]:

(a)  $F(X, V)$  is unique (up to isomorphism) if it exists (b)  $F(X, V)$  exists if and only if there is a member of  $V$  which has  $X$  as a subspace, (c)  $F(X, V)$  is the free group of the underlying variety of groups on the set  $X$ .

If  $V$  is a variety such that  $F(X, V)$  exists and is Hausdorff for each Tychonoff space  $X$ , then  $V$  is called a  $\beta$ -variety.

Using Świerczkowski [16] it is shown in [9] that every full variety is a  $\beta$ -variety. Finally we note that in [10] it is proved that  $\beta$ -varieties exist in profusion.

## § 2. The theorems.

**THEOREM 1.** *If  $\Omega$  is any class of arcwise connected Hausdorff groups and  $\Omega$  contains a non-trivial group then  $V(\Omega)$  is a  $\beta$ -variety.*

**Proof.** If  $X$  is any Tychonoff space, then Lemma 5, p. 116 of [6] implies that  $X$  can be embedded in a product of copies of any member of  $\Omega$ . Therefore  $F(X, V(\Omega))$  exists.

To show that  $F(X, V(\Omega))$  is Hausdorff it is sufficient to find for each element  $a \neq e_1$ , a member  $H$  of  $\Omega$  and a continuous homomorphism  $\Gamma$  of  $F(X, V(\Omega))$  into  $H$  such that  $\Gamma(a) \neq e$ , where  $e$  and  $e_1$  denote the identity elements of  $H$  and  $F(X, V(\Omega))$  respectively.

Suppose  $a = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$  where  $x_i \in X$  and  $\varepsilon_i$  is an integer for  $i = 1, \dots, n$ . Since  $a \neq e_1$ ,  $x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$  is not a law [15] of the underlying variety  $V(\Omega)$  of groups. Noting also that  $F(X, V(\Omega))$  is the free group on the set  $X$  of  $V(\Omega)$ , we see that there is a member  $H$  of  $\Omega$  and a (not necessarily continuous) homomorphism  $\varphi$  of  $F(X, V(\Omega))$  into  $H$  such that  $\varphi(x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}) \neq e$ .

Since  $X$  is a Tychonoff space and  $H$  is arcwise connected Hausdorff, Theorem 3.6 of [5] implies that there exists a continuous mapping  $\gamma$  of  $X$  into  $H$  such that  $\gamma(x_i) = \varphi(x_i)$  for  $i = 1, \dots, n$ . This implies that there exists a continuous homomorphism  $\Gamma$  of  $F(X, V(\Omega))$  into  $H$  such that  $\Gamma|X = \gamma$ . Clearly  $\Gamma(a) = \varphi(a) \neq e$  and the proof is complete.

If the "arcwise connected Hausdorff" condition is weakened to "connected Hausdorff" we expect, but have failed to prove, that the resulting proposition is false.

It would be interesting to know the answer to the following:

**OPEN QUESTION** (1). Can every  $\beta$ -variety be generated by its arcwise connected Hausdorff groups?

Two remarks are relevant. Firstly, we note that every  $\beta$ -variety contains non-trivial arcwise connected Hausdorff groups. This is easily seen by letting  $X$  be any arcwise connected Tychonoff space and observing that the subgroup of  $F(X, V)$  algebraically generated by  $x^{-1}X$ , for any  $x \in X$ , is an arcwise connected Hausdorff group. Secondly, we point out that it is clear from [4] that every full variety is generated by its arcwise connected Hausdorff groups.

(1) This question is answered in the negative in [2].

**THEOREM 2.** *If  $\Omega$  is any class of connected locally compact groups then  $V(\Omega)$  is a  $\beta$ -variety.*

*Proof.* It is clear from § 4.6 of [7] that any connected locally compact group  $H$  is a subgroup of a product of Lie groups, each of these Lie groups being a quotient of  $H$ . This implies that  $V(\Omega) = V(\Gamma)$ , where  $\Gamma$  is a class of connected Lie groups. Noting that any connected Lie group is arcwise connected, the result immediately follows from Theorem 1.

We note that the "connected" condition in the above theorem cannot be removed. For example, it is shown in [10] that a variety generated by discrete groups is not a  $\beta$ -variety.

It is (essentially) shown in [10] that any abelian variety containing a connected compact group is a  $\beta$ -variety. Example 3.2 of [10] shows that the "abelian" condition cannot be omitted. However our final theorem, which is a simple consequence of Theorem 3 of [1] and Lemma 4.8 of [10], is a companion to this.

**THEOREM 3.** *If the variety  $V$  contains a locally compact connected non-solvable group (in particular, a compact connected non-abelian group) then  $V$  is a  $\beta$ -variety.*

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