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FREE SUBGROUPS OF FREE TOPOLOGICAL GROUPS

David C. Hunt and Sidney A. Morris

SUMMARY

It is well known that every subgroup of a free group is a free group. However, it is not true in general that a subgroup of a free topological group is a free topological group.

It is shown that if X is the open unit interval $(0, 1)$ and Y is the closed unit interval $[0, 1]$ then the subgroup of the free topological group on Y generated by X is not a free topological group. On the other hand it is shown here that the commutator subgroup of the free topological group on Y is a free topological group.

1. Introduction

If G is a topological group then $|G|$ denotes the underlying abstract group and e always denotes the identity.

DEFINITION. If X is a topological space, with distinguished point e , the topological group $FG(X)$ is said to be the (Graev) *free topological group on* X *if*:

(a) $|FG(X)|$ is a free group with free basis $X\backslash\{e\}$ and identity e , and

(b) the topology of $FG(X)$ is the *finest* group topology on $|FG(X)|$ which induces the given topology on X .

NOTE. (i) Condition (b) can be replaced by (b'): for any continuous map γ of X into any topological group G such that $\gamma(e)$ is the identity element of G there exists a unique continuous homomorphism $\Gamma : FG(X) \to G$ such that $\Gamma|X = \gamma$.

(ii) $FG(X)$ is independent of the choice of the distinguished point e in X .

(iii) We define the free abelian topological group on X , $AG(X)$, by replacing "group" by "abelian group" in the above definition.

It is well known that every subgroup of a free group is a free group. However, it is not true in general that a subgroup of a free topological group is a free topological group. If X is compact and Y is a compact subspace of X , then the subgroup gp(Y) of $FG(X)$ generated algebraically by Y is the free topological group on Y ; (Graev [2]). A recent as yet unpublished result of R. Brown and L. Hardy is that if X is compact then any *open* subgroup of $FG(X)$ is a free topological group. However, Graev showed that a *closed* subgroup of a free topological group need not be a free topological group. Another example is:

EXAMPLE. $\text{gp}(0, 1) \subset FG[0, 1]$. Let $b_i = \frac{1}{\pi i}$ and $a_i = \frac{1}{(i+\frac{1}{2})\pi}$, $i = 1, 2, \ldots$, and let $e = \frac{1}{\pi}$. Now $\gamma(x) = \sin\left(\frac{1}{x}\right)$ is a continuous function from $(0, 1)$ to R such that $\gamma(e) = 0$, $\gamma(b_i) = 0$ and $\gamma(a_i) = 1$ for all i . We claim that γ cannot be extended to a continuous homomorphism $\Gamma : \text{gp}(0, 1) \to R$ and hence (by condition (b') above) $\text{gp}(0, 1) \neq FG(0, 1)$. Suppose that Γ exists, then, since Γ is continuous and $a_i b_i^{-1}$ approaches e and $\Gamma(e) = 0$, $\Gamma\left(a_i b_i^{-1}\right)$ approaches 0 . However, $\Gamma(a_i)\Gamma\left(b_i^{-1}\right) = 1$, so Γ is not a homomorphism.

This does *not* show that gp$(0, 1)$ is not a free topological group, merely that it is not $FG(0, 1)$. In §3 we develop sufficient theory to prove this new result. In §2 we prove that the commutator subgroup of $FG[0, 1]$ is a free topological group.

We complete this section by giving some definitions, notation and preliminary results. Elementary properties of topological groups are assumed and can be found in Bourbaki [1].

DEFINITION. A Hausdorff topological space X is said to be a k_ω*-space* if $X = \bigcup_{n=1}^{\infty} X_n$ where

(i) X_n is compact for all n ,

(ii) $X_n \subseteq X_{n+1}$, for all n ,

(iii) a subset A of X is closed in X if and only if $A \cap X_n$ is compact for each n .

When we say $X = \cup X_n$ is a k_ω-decomposition we mean that the X_n have the properties (i), (ii) and (iii).

As examples of k_ω-spaces we have any compact space and any connected locally compact group. Note that $(0, 1)$ is homeomorphic to the topological group R , of real numbers, and hence $(0, 1) = \cup\left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ is a k_ω-decomposition.

We will use the following properties of k_ω-spaces, (for further comments see

[5]):

(i) a closed subspace of a k_ω-space is a k_ω-space;

(ii) if Y is a compact subset of the k_ω-space $X = \cup X_n$, then there exists n such that $Y \subseteq X_n$;

(iii) if $X = \cup X_n$ is a k_ω-decomposition and $Y_1, Y_2, \ldots$ is a sequence of compact subsets of X such that each X_n is contained in some Y_m then $X = \cup Y_m$ is a k_ω-decomposition;

(iv) if G is a topological group and a k_ω-space then the X_n can be chosen such that $G = \cup X_n$ is a k_ω-decomposition and $X_n X_m \subseteq X_{n+m}$ for all n, m .

If G is a group and X is a subset of G , then $\mathrm{gp}(X)$ denotes the subgroup of G generated algebraically by X . Further $\mathrm{gp}_n(X)$ denotes the set of words in $\mathrm{gp}(X)$ of length less than or equal to n with respect to X .

THEOREM A [5]. *If $X = \cup X_n$ is a k_ω-space then $FG(X)$ is a k_ω-space with decomposition $FG(X) = \cup \mathrm{gp}_n(X_n)$.*

THEOREM B [5]. *Let $X = \cup X_n$ be a k_ω-space. Let $Y \subset FG(X)$ be a subset such that $Y \backslash \{e\}$ is a free algebraic basis for $\mathrm{gp}(Y)$. Suppose $Y_1, Y_2, \ldots$ is a sequence of compact subsets of Y such that $Y = \cup Y_n$ is a k_ω-decomposition of Y inducing the same topology on Y that Y inherits as a subset of $FG(X)$. Put $X^n = \mathrm{gp}_n(X_n)$ and $Y^n = \mathrm{gp}_n(Y_n)$. If for each natural number n there is an m such that $\mathrm{gp}(Y) \cap X^n \subseteq Y^m$, then $\mathrm{gp}(Y)$ is the free topological group on Y and both $\mathrm{gp}(Y)$ and Y are closed subsets of $FG(X)$.*

We note that the family $\{F_\alpha : \alpha \in I\}$ of sets is said to have the *finite intersection property* if for each $\alpha_1, \ldots, \alpha_n \in I$, $F_{\alpha_1} \cap \ldots \cap F_{\alpha_n} \neq \emptyset$. If X is a subset of a topological space, $\bar{X}$ denotes the closure of X .

DEFINITION. A *filterbase* on a set X is a set $\underline{\underline{F}}$ of subsets of X such that

(i) every finite intersection of sets in $\underline{\underline{F}}$ is a member of $\underline{\underline{F}}$;

(ii) $\emptyset \notin \underline{\underline{F}}$ and $\underline{\underline{F}}$ is non-empty.

DEFINITION. Let X be a topological space and $\underline{\underline{F}}$ a filterbase on X then $\underline{\underline{F}}$ is said to be *convergent* if there exists x in X such that every neighbourhood of x contains an element of $\underline{\underline{F}}$.

DEFINITION. A filterbase $\underline{\underline{F}}$ on a topological group G is said to be a *Cauchy filterbase* (in the left uniformity) if for each neighbourhood U of e there exists F in $\underline{\underline{F}}$ such that $F^{-1}F \subseteq U$.

DEFINITION. A topological group is said to be *complete* if every Cauchy filterbase is convergent.

PROPOSITION [1]. *If G is a topological group and H is a subgroup of G which is complete then H is closed in G.*

2. The commutator subgroup

THEOREM 1. *The commutator subgroup of $FG([0, 1])$ is a free topological group.*

PROOF. Let F' be the commutator subgroup. Then F' has a free basis equal to $Y\backslash\{e\}$, where

$$Y = \left\{x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n} x_i x_n^{-\varepsilon_n} \ldots x_i^{-\varepsilon_i - 1} \ldots x_1^{-\varepsilon_1} \mid \varepsilon_i \text{ integers}, \ x_i \in [0, 1] , \ i = 1, \ldots, n \text{ and } x_1 \le x_2 \le \ldots \le x_n\right\} ; \quad \text{(see [4])}.$$

We will show that F' is the free topological group on Y.

Put $Y_m = Y \cap \text{gp}_m[0, 1]$ and $Y^m = \text{gp}_m(Y_m)$. By Theorem B of §1 it suffices to show

(a) for each n, there exists m such that $\text{gp}(Y) \cap \text{gp}_n[0, 1] \subseteq Y^m$, and

(b) Y_m is compact.

Let $F\{x_1, \ldots, x_n\}$ be the algebraic free group on $x_1, \ldots, x_n$. Define

$$Z = \left\{x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n} x_i x_n^{-\varepsilon_n} \ldots x_i^{-\varepsilon_i - 1} \ldots x_1^{-\varepsilon_1} \mid \varepsilon_i \text{ integers}, \ x_i \in [0, 1] , \ i = 1, \ldots, n\right\} .$$

Clearly $\text{gp}(Z) \cap \text{gp}_n\{x_1, \ldots, x_n\}$ is finite and therefore is contained in $\text{gp}_m(Z)$ for some m.

Now let $w \in \text{gp}(Y) \cap \text{gp}_n[0, 1]$.

Then there exists $x_1, x_2, \ldots, x_n$ in $[0, 1]$, $x_i \ne e$, $i = 1, 2, \ldots, n$, and $x_1 \le x_2 \le \ldots \le x_n$ such that $w \in \text{gp}_n\{x_1, \ldots, x_n\}$. By the comment in the above paragraph there exists m such that $w \in \text{gp}_m(Y)$. Since m is independent of

$x_1, \ldots, x_n$ we have (a) is true. Y_m is a *finite* union of sets

$$Y_{n,\varepsilon} = \left\{x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n} x_i x_n^{-\varepsilon_n} \ldots x_i^{-\varepsilon_i - 1} \ldots x_1^{-\varepsilon_1} \text{ where } x_i \in [0, 1] , \right.$$
$$\left. x_1 \le x_2 \le \ldots \le x_n \text{ for a fixed } n \text{ and fixed } \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)\right\} .$$

To prove (b) it suffices to show each $Y_{n,\varepsilon}$ is compact. Now $Y_{n,\varepsilon}$ is a continuous image of the set

$$K_n = \{(x_1, x_2, \ldots, x_n) \mid x_1 \le x_2 \le \ldots \le x_n\}$$
$$\subseteq [0, 1] \times [0, 1] \times \ldots \times [0, 1] \quad (n \text{ copies of } [0, 1])$$

under the obvious map f_ε . Clearly each K_n is compact. Hence $Y_{n,\varepsilon}$ is compact as required.

REMARK 1. Theorem 1 is also true if $[0, 1]$ is replaced by any totally ordered topological space which is k_ω in the order topology.

REMARK 2. On the other hand F. Clarke in a private communication stated that the commutator subgroup of $FG(S^n)$ is not a free topological group where S^n is the n sphere.

REMARK 3 . Theorem 1 is also true if the commutator subgroup is replaced by any verbal subgroup V_m containing the commutator subgroup. Of course V_m is the verbal subgroup defined by the words $[x, y]$ and x^m . The only difference in the proof is that Y is replaced by

$$Y(m) = \left\{x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n} x_i x_n^{-\varepsilon_n} \ldots x_i^{-\varepsilon_i - 1 + \delta} \ldots x_1^{-\varepsilon_1} \mid 0 \le \varepsilon_i \le m , \ x_i \in [0, 1] , \right.$$
$$i = 1, \ldots, n, \ x_1 \le x_2 \le \ldots \le x_n \text{ and } \delta = m \text{ if } \varepsilon_i = m - 1 \text{ and}$$
$$\left. \delta = 0 \text{ otherwise}\right\} .$$

This suggests the natural question:

QUESTION 1. *Which verbal subgroups and which characteristic subgroups of* $FG[0, 1]$ *are free topological groups?*

3. Completeness of k_ω-groups and its consequences

The following theorem generalizes Theorem 6 of Graev [2]. The proof given is essentially the same as Graev's.

THEOREM 2. *If* G *is a* k_ω *group then it is complete.*

PROOF. Let $G = \cup X_n$ be a k_ω-space decomposition of G such that $X_n X_m \subseteq X_{n+m}$

and $e \in X_1$.

Suppose G is not complete. Then there exists a Cauchy filterbase $\underline{\underline{F}}'$ in G which does not converge. Put $\underline{\underline{F}} = \{F_\alpha \mid \alpha \in I\}$ where each F_α is a finite intersection of closures of elements of $\underline{\underline{F}}'$. Clearly $\underline{\underline{F}}$ is a Cauchy filterbase which does not converge. Now $\bigcap_{\alpha \in I} F_\alpha = \emptyset$. For if $x \in \bigcap_\alpha F_\alpha$, let U be a neighbourhood of x then $x^{-1}U$ is a neighbourhood of e and $x^{-1}U \supseteq F_\alpha^{-1}F_\alpha \supseteq x^{-1}F_\alpha$, for some α , and hence $U \supseteq F_\alpha$ and the filterbase converges to x , a contradiction.

If $\{X_n \cap F_\alpha \mid \alpha \in I\}$ had the finite intersection property then, since X_n is compact, $\bigcap_\alpha \left(X_n \cap F_\alpha\right) \neq \emptyset$ which is false. Hence $\{X_n \cap F_\alpha \mid \alpha \in I\}$ does not have the finite intersection property. Thus for all $n \geq 2$ there exists $\alpha_n \in I$ such that $X_{n-1} \cap F_{\alpha_n} = \emptyset$.

We shall construct a sequence of sets U_i $(i = 1, 2, \ldots)$ in the group G with the following properties:

(a) $e \in U_1$;

(b) $U_i \subseteq X_i$ and U_i is open in X_i ;

(c) $U_j \subseteq U_i$ if $j \leq i$;

(d) $X_j\bar{U}_i \cap F_{\alpha_{2j}} = \emptyset$ if $j \leq i$.

Before continuing the proof we need a lemma.

LEMMA. *If M is a compact subset of G and N is a closed subset of G and $M \cap N = \emptyset$ then there exists an open neighbourhood U of e such that $M\bar{U} \cap N = \emptyset$.*

PROOF. For each $m \in M$ there exists an open neighbourhood $V(m)$ of e such that $mV^2(m) \cap N = \emptyset$. Since M is compact and $M \subseteq \bigcup_{m \in M} mV(m)$ then

$$M \subseteq \bigcup_{i=1}^{n} m_iV\left(m_i\right) , \quad m_i \in M .$$

Put $V = \bigcap_{i=1}^{n} V\left(m_i\right)$. Then $MV \cap N = \emptyset$. Using the regularity of a topological group we can see that there exists an open neighbourhood U of e such that $\bar{U} \subset V$ and hence $M\bar{U} \cap N = \emptyset$, as required.

Taking $M = X_1$ and $N = F_{\alpha_2}$ the lemma gives an open neighbourhood $U^{(1)}$ of e

such that $X_1\overline{U^{(1)}} \cap F_{\alpha_2} = \emptyset$. Put $U_1 = U^{(1)} \cap X_1$. Then (a), (b), (c), (d) are satisfied for U_1 . Suppose $U_1, \ldots, U_n$ have been constructed with the required properties. We proceed to construct U_{n+1} .

By (d), $X_j\bar{U}_n \cap F_{\alpha_{2j}} = \emptyset$, $j = 1, \ldots, n$. Furthermore it is evident that $X_{n+1}\bar{U}_n \cap F_{\alpha_{2(n+1)}} = \emptyset$ as $X_{n+1}\bar{U}_n \subseteq X_{n+1}X_n \subseteq X_{2n+1}$.

Since $X_j\bar{U}_n$ is compact for $j = 1, \ldots, n+1$ there exists an open neighbourhood V of e such that $X_j\bar{U}_n\bar{V}_n \cap F_{\alpha_{2j}} = \emptyset$, $j = 1, \ldots, n+1$. Put $U_{n+1} = U_n.V \cap X_{n+1}$. Clearly U_{n+1} satisfies conditions (a), (b), (c) and since $U_{n+1} \subseteq \bar{U}_n.\bar{V}$ we have (d) for U_{n+1} .

Now set $U = \bigcup_{i=1}^{\infty} U_i$. Since G is a k_ω-space U is open in G . From condition (d), $X_jU \cap F_{\alpha_{2j}} = \emptyset$ for all j .

Since $\{F_\alpha\}$ is a Cauchy filterbase and U is an open neighbourhood of e , there exists F_α such that $F_\alpha^{-1}F_\alpha \subseteq U$. Let $x \in F_\alpha$ so $F_\alpha \subseteq xU$. Now $x \in X_n$ for some n and therefore $xU \cap F_{\alpha_{2n}} = \emptyset$ and $F_\alpha \cap F_{\alpha_{2n}} = \emptyset$, a contradiction. The proof is complete.

Since a free topological group on a k_ω-space is a k_ω-space (Theorem A) we have

COROLLARY 1. *If X is a k_ω-space then $FG(X)$ is a complete topological group.*

As an interesting application of Theorem 2 we prove the converse of Theorem B.

THEOREM 3. *Let $X = \cup X_n$ be a k_ω-space and $Y = \cup Y_m$ be a k_ω-space $\subseteq FG(X)$. Then* $\mathrm{gp}(Y) = FG(Y)$ *if and only if for each n there exists an m such that* $\mathrm{gp}(Y) \cap X^n \subseteq Y^m$ *and $Y\backslash\{e\}$ is a free algebraic basis for* $\mathrm{gp}(Y)$.

PROOF. Suppose that there exists n such that for no m is $\mathrm{gp}(Y) \cap X^n \subseteq Y^m$. Then there exists a sequence $\{a_m\}$ of elements of $\mathrm{gp}(Y) \cap X^n$ such that $a_m \notin Y^m$, any m . Let $S = \{a_n\}$. Then $S \cap Y^m$ is compact (as it is finite) for each m . By Theorem A, $FG(Y) = \cup Y^m$ is a k_ω-decomposition. So S is closed in $FG(Y)$. Also $S - \{a_n\}$ for any n is closed in $FG(Y)$ and hence in S . Thus S has the discrete topology. Since $FG(Y)$ is a k_ω-space it is complete and therefore closed in $FG(X)$. So S is a closed infinite discrete subset of X^n, which is impossible as X^n is compact. The converse is Theorem B.

The following Corollary extends Theorem 7 of Graev [2].

COROLLARY 2. *Let* $X = \cup X_n$ *and* $Y = \cup Y_n$ *be* k_ω*-spaces and suppose* $FG(X) = FG(Y)$. *If each* X_n *is metric, then each* Y_n *is metric.*

We will need some information on the following question:

Is the closure of a free topological group necessarily a free topological group? (More precisely if $FG(X)$ *is a subgroup of a topological group* G, *is* $\overline{FG(X)}$ *a free topological group? Also is* $\mathrm{gp}(\bar{X})$ *a free topological group?)* As a technical proposition in this direction we have:

PROPOSITION 1. *If* $X \subset G$ *and* $\mathrm{gp}(X) = FG(X)$ *then* $\mathrm{gp}(\bar{X})$ *has the property that if* H *is any topological group with a completion and* ϕ *is any continuous map* $\bar{X} \to H$ *such that* $\phi(e) = e$ *then there exists a continuous homomorphism* $\Gamma : \mathrm{gp}(\bar{X}) \to H$ *such that* $\Gamma|_{\bar{X}} = \phi$.

PROOF. Firstly $\mathrm{gp}(X)$ is dense in $\mathrm{gp}(\bar{X})$. For if $a = x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n} \in \mathrm{gp}(\bar{X})$ and if U is a neighbourhood of a then there exist neighbourhoods $V_1, \ldots, V_n$ of $x_1, \ldots, x_n$ in G such that $V_1^{\varepsilon_1} V_2^{\varepsilon_2} \ldots V_n^{\varepsilon_n} \subseteq U$ and there exist elements $z_i \in V_i \cap X$ such that $z_1^{\varepsilon_1} z_2^{\varepsilon_2} \ldots z_n^{\varepsilon_n} \in U$.

Now $\phi|_X$ is continuous : $X \to H$ and $\phi(e) = e$. Hence there exists a continuous homomorphism $\Phi : FG(X) = \mathrm{gp}(X) \to H$ such that $\Phi|_X = \phi$. Let the topological group $\hat{H}$ be the completion of H. Since $FG(X)$ is dense in $\mathrm{gp}(\bar{X})$, there exists a continuous homomorphism $\Gamma : \mathrm{gp}(\bar{X}) \to \hat{H}$ such that $\Gamma|FG(X) = \Phi$, [1, page 246, Proposition 5]. Clearly $\Gamma|\bar{X} = \phi$. But $\Gamma(\bar{X}) \subseteq H$ and therefore $\Gamma(\mathrm{gp}(X)) \subseteq H$.

As every abelian topological group has a completion we have:

COROLLARY 3. *If* $X \subseteq G$ *and* $\mathrm{gp}(X) = AG(X)$ *then* $\mathrm{gp}(\bar{X}) = AG(\bar{X})$.

COROLLARY 4. *If* $X \subset FG(Y)$, *where* Y *is a* k_ω*-space, and* $\mathrm{gp}(X) = FG(X)$ *then* $\mathrm{gp}(\bar{X}) = FG(\bar{X})$.

PROOF. Firstly note that by Corollary 1, $FG(\bar{X})$ is a complete topological group. Consider the identity $\phi : \bar{X} \left(\subseteq \mathrm{gp}(\bar{X})\right) \to \bar{X} \left(\subseteq FG(\bar{X})\right)$. By Proposition 1, there exists a continuous homomorphism $\Gamma : \mathrm{gp}(\bar{X})\ (\subseteq G) \to FG(\bar{X})$ such that $\Gamma|\bar{X} = \phi$.

Since $|FG(\bar{X})|$ is the free group on $\bar{X}\backslash\{e\}$, $|\mathrm{gp}(\bar{X})|$ must be the free group on $\bar{X}\backslash\{e\}$ and Γ must be the identity map. Since Γ is continuous and $FG(\bar{X})$ has the finest group topology with respect to $\bar{X}$, Γ must be a homeomorphism; that is, $\mathrm{gp}(\bar{X}) = FG(\bar{X})$.

NOTE. Y is a k_ω-space would be an unnecessary restriction if it could be shown that every free topological group has a completion.

QUESTION 2. *Does every free topological group have a completion?*

As an extension of the Example in §1 we show:

THEOREM 4. *If* X *is metric and* $Y \subset X \subset FG(X)$ *then* $\mathrm{gp}(Y) = FG(Y)$ *implies* Y *is closed in* X .

PROOF. Suppose not and let a belong to $\bar{Y}\backslash Y$. Let $\{a_n\}$, $\{b_n\}$ be sequences in Y such that

(i) $a_n \to a$,

(ii) $b_n \to a$,

(iii) $a_n \neq b_m$ for all n and m ,

(iv) $b_n \neq e$, $a_n \neq e$ for all n .

Put $S = \{a_n \mid n = 1, \ldots\} \cup \{b_n \mid n = 1, \ldots\} \cup \{e\}$. Define a continuous function $\phi : S \to R$ by

$$\phi(a_n) = 1 \text{ for all } n ,$$

$$\phi(b_n) = 0 \text{ for all } n ,$$

$$\phi(e) = 0 .$$

Since Y is normal and S is closed in Y by the Tietze Extension Theorem, ϕ can be extended to a countinuous map $\phi : X \to R$.

However ϕ cannot be extended to a continuous homomorphism $\Phi : \mathrm{gp}(Y) \to R$ as $a_n b_n^{-1} \to e$, implies $\Phi\left(a_n b_n^{-1}\right)$ would converge to 0 while $\Phi(a_n)\Phi\left(b_n^{-1}\right) = 1$.

Hence $Y = \bar{Y}$ and Y is closed in X .

THEOREM 5. *The subgroup* $\mathrm{gp}(0, 1)$ *of* $FG[0, 1]$ *is not a free topological group.*

PROOF. Suppose $\mathrm{gp}(0, 1) = FG(X)$ for some topological space X . Then by Corollary 4, $\mathrm{gp}(\bar{X}) = FG(\bar{X})$. By Theorem A, $FG[0, 1]$ is a k_ω-group and therefore the closed subset $\bar{X}$ is a k_ω-space. Therefore by Corollary 1, $FG(\bar{X})$ is complete and hence closed in $FG[0, 1]$. However $FG(\bar{X}) \supseteq \mathrm{gp}(0, 1)$ and $\mathrm{gp}(0, 1)$ is dense in $FG[0, 1]$. Therefore $FG(\bar{X}) = FG[0, 1]$. By Theorem 7 of Graev [2], $\bar{X}$ is compact and metrizable. Applying Theorem 4 with Y and X replaced by X and $\bar{X}$ respectively yields X is closed in $\bar{X}$; that is, $X = \bar{X}$. Hence $\mathrm{gp}(0, 1) = FG(X)$ and X is compact.

Therefore by Thoerem A, $\mathrm{gp}(0, 1)$ is a k_ω-group and hence by Theorem 2 is complete. Therefore $\mathrm{gp}(0, 1)$ is closed in $FG[0, 1]$, which is clearly false.

REMARK. The above theorem holds with $[0, 1]$ replaced by any compact metric space and $(0, 1)$ replaced by any dense subset.

QUESTION 3. *If* $Y \subset FG(X)$ *and* $\mathrm{gp}(Y) = FG(Y)$, *does this imply that*

(i) Y *is closed in* $FG(X)$ *?*

(ii) $FG(Y)$ *is closed in* $FG(X)$ *?*

(Note: *(ii)* implies *(i)*.)

Theorem 6 provides an answer in a special case.

THEOREM 6. *If* $X = \cup X_n$ *is metric and a* k_ω*-space and* $Y \subset X^n \subset FG(X)$ *then* $\mathrm{gp}(Y) = FG(Y)$ *implies* Y *is closed in* $FG(X)$. *Consequently* Y *is a* k_ω*-space and* $FG(Y)$ *is closed in* $FG(X)$.

PROOF. By Corollary 4, $\mathrm{gp}(\bar{Y}) = FG(\bar{Y})$. Noting that $\bar{Y} \subset X^n$ and X^n is metric we see that $\bar{Y}$ is metric. So Y is metric and $Y \subset \bar{Y} \subset FG(\bar{Y})$ and $\mathrm{gp}(Y) = FG(Y)$, hence by Theorem 4, Y is closed in $\bar{Y}$, that is, $Y = \bar{Y}$ as required.

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University of New South Wales,
Kensington, NSW 2033.