J. Group Theory **14** (2011), 931–935
DOI 10.1515/JGT.2011.090

**Journal of Group Theory**


# Local splitting of locally compact groups and pro-Lie groups

Karl H. Hofmann and Sidney A. Morris

(Communicated by George Willis)

**Abstract.** In the book "The Lie Theory of Connected Pro-Lie Groups" the authors proved the local splitting theorem for connected pro-Lie groups. George A. A. Michael subsequently proved this theorem for almost connected pro-Lie groups. Here his result is proved more directly using the machinery of the aforementioned book.

## Historical background

A significant portion of the research on topological groups done in the first half of the twentieth century was related to Hilbert's Fifth Problem in 1900 asking whether a locally euclidean topological group admits a Lie group structure. En route to the solution in 1952 by D. Montgomery, L. Zippin and A. Gleason there were remarkable contributions by H. Weyl, L. S. Pontryagin, E. R. van Kampen and J. von Neumann. The solution to Hilbert's Fifth Problem was a signpost on the road to understanding that locally compact groups are locally projective limits of Lie groups which was completed by H. Yamabe [10]. This information was the crucial hypothesis by K. Iwasawa [7] in 1949 which provided the thrust for half a century more research on the structure of locally compact groups. One of the key results is the following:

**Theorem.** *A* connected *locally compact group $G$ has arbitrarily small normal subgroups $N$ for each of which there is a local Lie group $C_N \subseteq G$ such that*

$$(n, c) \mapsto nc : N \times C_N \to NC_N \subseteq G$$

*is an isomorphism onto an identity neighborhood of $G$. In particular, if $L_N$ denotes the simply connected Lie group which is locally isomorphic with $C_N$, then the topological groups $G$, $N \times L_N$, and $N \times G/N$ are locally isomorphic.*

## Status

In the course of time one learned that the hypothesis of connectivity on $G$ could be dropped; the current status of the *local splitting theorem* for locally compact groups is therefore the following:

**Local splitting theorem for locally compact groups.** *Any locally compact group G has arbitrarily small subgroups N of G for each of which there is a connected Lie group $L_N$ and an injective morphism $\alpha : L_N \to G$ such that $(n, x) \mapsto n\alpha(x) : N \times L_N \to G$ is an open morphism with discrete kernel. In particular,* $G$, $N \times L_N$, *and* $N \times G_0N/N$ *are locally isomorphic*.

This result is recorded and proved in [3, Theorem 4.1, p. 474]. Note that the subgroups $N$ in that theorem will generally be normal only in some open subgroup of $G$, in accordance with the fact that $G$ may not have arbitrarily small normal subgroups. Yet small normal co-Lie subgroups exist if $G$ is almost connected, that is, if $G/G_0$ is compact where $G_0$ denotes the identity component.

Yamabe's Theorem, which says that a locally compact, almost connected group $G$ has arbitrarily small normal subgroups $N$ such that $G/N$ is a Lie group, is (in the presence of local compactness) equivalent to the property that $G$ is a projective limit of (finite dimensional real) Lie groups. A projective limit of finite dimensional real Lie Groups is known as a pro-Lie group. Equivalently, a *pro-Lie group* $G$ is a topological group which is complete in the sense of A. Weil and in which the set $\mathcal{N}$ of normal subgroups $N$ such that $G/N$ is a Lie group is a filter basis converging to 1. We call $\mathcal{N}$ the *standard filter basis* of $G$.

In view of Yamabe's Theorem, every locally compact group has an open subgroup which is a pro-Lie group. In other words the class of pro-Lie groups generalizes almost connected (and to a certain extent all) locally compact groups. One should bear in mind, however, that groups like $\mathrm{SL}(2, \mathbb{Q}_p)$ are locally compact but fail to be pro-Lie groups.

In view of this background it is a legitimate question to ask whether a pro-Lie group, in the absence of local compactness, still allows a *local splitting theorem*. This is not the case: It is observed in [4, Example 14.23, p. 600f.] that local splitting does not apply even to connected pro-Lie groups in general.

A comprehensive theory of *connected* pro-Lie groups is presented in [4]. Its main point is that an effective Lie theory is available in the sense that each pro-Lie group has an associated pro-Lie algebra, that is, a topological Lie algebra which is a projective limit of finite dimensional real Lie algebras. The structure theory of pro-Lie algebras is discussed in detail in [4]; it is simpler and more explicit than that of pro-Lie groups. The philosophy therefore is to use the pro-Lie algebra $\mathfrak{g}$ of a connected pro-Lie group $G$ to derive structural information on $G$. A theory of local splitting for pro-Lie groups was presented in [5] and [4]. It was shown that a sufficient condition for a Local Splitting Theorem on a connected pro-Lie group $G$ can be read off its Lie algebra $\mathfrak{g}$ as follows: First recall that a pro-Lie algebra is said to be *pronilpotent* if every finite dimensional homomorphic image is nilpotent. (See [4, Definition 7.42, p. 289 and Theorem 7.66, p. 303].) There exists a largest pronilpotent ideal $\mathfrak{n}$, called the *nilradical* in [4]. Let $\mathfrak{z}$ denote the center of $\mathfrak{g}$. We recall that a connected pro-Lie group is called *reductive* if the radical (that is, the largest prosolvable ideal) is central. In the following $\mathfrak{r}$ denotes this radical. Now we have:

**Local splitting theorem for connected pro-Lie groups** ([4, Theorem 3.19, p. 581]). *For a connected pro-Lie group G the following conditions are equivalent*:

(i) $\dim \mathfrak{r}/\mathfrak{z} < \infty$.

(ii) *$G$ is locally isomorphic to the direct product of a closed normal almost connected subgroup $N$ with reductive identity component $N_0$ and a connected Lie group $L$.*

We remark that G. A. A. Michael correctly points out in [9] that the result by Hofmann and Morris in [4, 13.17] and in [5, 3.2] is proved for *connected* pro-Lie groups only, motivating the extension to almost connected groups.

## A structure theorem on pro-Lie algebras

For a pro-Lie algebra $\mathfrak{g}$ let $\mathfrak{n}$ be the nilradical and $\mathfrak{s}$ the *coreductive radical*, that is, the smallest closed ideal such that $\mathfrak{g}/\mathfrak{s}$ is reductive. (In [4] Hofmann and Morris denote the coreductive radical of $\mathfrak{g}$ by $\mathfrak{n}_{\mathrm{cored}}$.) Here a pro-Lie algebra is *reductive* iff it is a product of a family of simple Lie algebras or copies of $\mathbb{R}$. Regarding local splitting, the following is the key result on the pro-Lie algebra level:

**Theorem 1.** *For a pro-Lie algebra $\mathfrak{g}$ with radical $\mathfrak{r}$, the following conditions are equivalent*:

(i) $\dim \mathfrak{n}/\mathfrak{z} < \infty$.

(ii) *$\mathfrak{g}$ is an ideal direct sum, algebraically and topologically, of a reductive pro-Lie algebra and a finite-dimensional Lie algebra.*

(iii) $\dim \mathfrak{s} < \infty$.

(iv) $\dim \mathfrak{r}/\mathfrak{z} < \infty$.

*Proof.* The equivalence of (i) and (ii) is [4, Theorem 13.15, p. 577].

Trivially (ii) implies (iii). Conversely, assume (iii), i.e., that $\mathfrak{s}$ is finite dimensional. By [4, Theorem 7.67, p. 304] $\overline{[\mathfrak{r}, \mathfrak{g}]} = \mathfrak{s}$ and thus $(x, y) \mapsto [x, y] : \mathfrak{r} \times \mathfrak{g} \to \mathfrak{s}$ is a continuous bilinear map into a finite dimensional vector space. In view of [4, Lemma A2.21, p. 646] there exists a closed vector subspace $E$ of $\mathfrak{r}$ such that $[E, \mathfrak{g}] = \{0\}$ and $\dim \mathfrak{r}/E < \infty$. Then $E \subseteq \mathfrak{z} \subseteq \mathfrak{r}$ and so $\dim \mathfrak{r}/\mathfrak{z} < \infty$, that is, (iv). Trivially, (iv) implies (i). Thus Theorem 1 is proved. □

The equivalence of (iii) and (i) in Theorem 1 is a reformulation of a result with a substantially different proof by George A. A. Michael [9].

## A local splitting theorem for almost connected pro-Lie groups

Now we prove the following result of G. A. A. Michael:

**Theorem 2** (Local splitting theorem for almost connected pro-Lie groups). *If $G$ is an almost connected pro-Lie group such that the coreductive radical $\mathfrak{s}$ of $\mathfrak{g}$ is finite dimensional, then there are arbitrarily small normal subgroups $N$ such that there is a simply connected Lie group $L_N$ and a morphism $\alpha : L_N \to G$ such that*

$(n, x) \mapsto n\alpha(x) : N \times L_N \to G$ *is open and has discrete kernel. In particular,* $G$ *and* $N \times L_N$ *are locally isomorphic.*

*Proof.* Let $N \in \mathcal{N}$ where $\mathcal{N}$ is the standard filter basis of $G$. Then $G_0 N/N$ is the identity component of the Lie group $G/N$ (cf. [4, Lemma 3.29(ii), p. 152]), whence $G_0 N$ is an open (hence closed) subgroup of $G$ and therefore is a pro-Lie subgroup. The normal subgroup $M = G_0 \cap N$ provides a bijective morphism $G_0/M \to G_0 N/N$ which is an isomorphism, provided $N$ is chosen small enough [6]. Therefore, every pro-Lie group $G$ has arbitrarily small $N \in \mathcal{N}$ such that $G_0 N$ is an open subgroup of $G$ and that $G_0 N/N \cong G_0/M$ is a connected Lie group. By the local splitting theorem for connected pro-Lie groups, $N$ can be chosen so that there is a simply connected Lie group $L_N$ and a morphism $\alpha : L_N \to G_0$ such that $(m, x) \mapsto m\alpha(x) : M \times L_N \to G_0$ is an open surjective morphism with discrete kernel. The action of $L_N$ on $N$ via $x \cdot n = \alpha(x) n \alpha(x)^{-1}$ permits the formation of the semidirect product $N \rtimes L_N$ and an open morphism $(n, x) \mapsto n\alpha(x) : N \rtimes L_N \to G$ with a discrete kernel and the image $G_0 N$. For a proof of the local splitting theorem we may just as well assume that $G$ is the semidirect product $NL$ of $N$ and a simply connected closed Lie subgroup $L$ and that the product $ML$ is direct. Then, on the Lie algebra level, $\mathfrak{g}$ is the ideal direct sum $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{l}$; in particular, $\mathfrak{l}$ is a finite dimensional ideal. The adjoint representation of $G$ provides us with an automorphic action $(g, X) \mapsto g \cdot X : G \times \mathfrak{g} \to \mathfrak{g}$ of the group $G$ so that $\exp g \cdot X = g(\exp X)g^{-1}$. We note that

$$M \text{ fixes } \mathfrak{l} \text{ elementwise.} \tag{1}$$

Assuming that the equivalent conditions of Theorem 1 are satisfied by $\mathfrak{g}$, we see that by choosing $N$ sufficiently small, the following condition holds:

$$\mathfrak{m} \text{ is a reductive ideal.} \tag{2}$$

Now let $Z(M, G_0)$ be the centralizer of $M$ in $G_0$. Since $M$ is normal in $G$ so is $Z(M, G_0)$. Its Lie algebra is the centralizer $\mathfrak{z}(\mathfrak{m}, \mathfrak{g})$ (see [4, Proposition 9.17(ii), p. 371]). Note that $Z(M, G_0)$ contains $L$ and all conjugates $nLn^{-1}$, $n \in N$. Accordingly, the conjugates $n \cdot \mathfrak{l}$, $n \in N$ are contained in $\mathfrak{z}(\mathfrak{m}, \mathfrak{g})$. Moreover, $\mathfrak{z}(\mathfrak{m}, \mathfrak{g})$ is the ideal direct sum $(\mathfrak{z} \cap \mathfrak{m}) \oplus \mathfrak{l}$ where $\mathfrak{z} = \mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$. Since $M$ fixes $\mathfrak{z}(\mathfrak{m}, \mathfrak{g})$ elementwise by (1), we have an action of $N/M$ on $\mathfrak{z}(\mathfrak{m}, \mathfrak{g})$ given by $(nM) \cdot X = n \cdot X$. The action of $N/M$ leaves the commutator algebra $\mathfrak{z}(\mathfrak{m}, \mathfrak{g})' = \mathfrak{l}'$ invariant. Thus the complete topological vector space $\mathfrak{u} \stackrel{\text{def}}{=} \mathfrak{z}(\mathfrak{m}, \mathfrak{g})/\mathfrak{l}'$ is an $N/M$-module and $\mathfrak{v} \stackrel{\text{def}}{=} \mathfrak{z} \cap \mathfrak{m}$ is a closed submodule.

Now let us provisionally assume that there is a module complement $\mathfrak{w}$ of $\mathfrak{v}$ in $\mathfrak{u}$. Then the full inverse image $\mathfrak{l}^*$ of $\mathfrak{w}$ in $\mathfrak{z}(\mathfrak{m}, \mathfrak{g})$ is an ideal direct summand of $\mathfrak{g}$ which is invariant under the action of $N$. The projection $G \to L \cong G/N$ maps the analytic subgroup $L^* \stackrel{\text{def}}{=} \langle \exp \mathfrak{l}^* \rangle$ onto $L$; since $L$ is simply connected, $L^* \cong L$, and $G = NL^*$ is a direct product, and this conclusion will complete the proof of the theorem.

Thus the proof of the theorem depends on the splitting of the $N/M$-submodule $\mathfrak{v}$ of $\mathfrak{u}$. However, if $N/M$ can be recognized as being compact, then, as an $N/M$-

module, $\mathfrak{u}$ is semisimple and therefore the submodule $\mathfrak{v}$ does have an invariant complement.

However, by our assumption, $G = NL$, and this implies $G_0 = N_0L$, and therefore $G_0N = N(N_0L) = G$ whence $G/G_0 = (G_0N)/G_0 \cong N/M$ and $N/M$ thus, finally, the assumed compactness of $G/G_0$ implies that of $N/M$ which we need to complete the proof. □

**Corollary 3.** *Assume that $G$ is a pro-Lie group such that $G/G_0$ is locally compact and that the coreductive radical $\mathfrak{s}$ of $\mathfrak{g}$ is finite dimensional. Then the local splitting theorem applies to $G$.*

*Proof.* For every pro-Lie group $G$, the factor group $G/G_0$ has arbitrarily small open normal subgroups (see [4, Proposition 3.30(b)]). Thus, if $G/G_0$ is locally compact, there are arbitrarily small compact open normal subgroups $G_1/G_0$ of $G/G_0$. The assertion then follows from Theorem 2. □

## References

[1] N. Bourbaki. *Groupes et algèbres de Lie* (Hermann, 1971).
[2] K. H. Hofmann and S. A. Morris. *The Structure of Compact Groups*, 2nd Edition (Verlag Walter De Gruyter, 2006).
[3] K. H. Hofmann and S. A. Morris. Transitive actions of compact groups and topological dimension. *J. Algebra* **234** (2000), 454–479.
[4] K. H. Hofmann and S. A. Morris. *The Lie Theory of Connected Pro-Lie Groups* (EMS Publishing House, 2007).
[5] K. H. Hofmann and S. A. Morris. Iwasawa's local splitting theorem for pro-Lie groups. *Forum Math.* **20** (2008), 607–629.
[6] K. H. Hofmann and S. A. Morris. On the pro-Lie Group theorem and the closed subgroup theorem. *J. Lie Theory* **18** (2008), 383–390.
[7] K. Iwasawa. On some types of topological groups. *Ann. of Math.* **50** (1949), 507–558.
[8] G. A. A. Michael. A Counterexample in the Dimension Theory of Homogeneous Spaces of Locally Compact Groups. *J. Lie Theory* **18** (2008), 915–917.
[9] G. A. A. Michael. The generalized Gluškov–Iwasawa local splitting theorem. *J. Math. Sci. Univ. Tokyo* **17** (2010), 123–133.
[10] H. Yamabe. On the conjecture of Iwasawa and Gleason. *Ann. of Math.* **58** (1953), 48–54.
[11] H. Yamabe. Generalization of a theorem of Gleason. *Ann. of Math.* **58** (1953), 351–365.

Received 20 June, 2010; revised 23 December, 2010

Karl H. Hofmann, Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstrasse 7, 64289 Darmstadt, Germany
E-mail: hofmann@mathematik.tu-darmstadt.de

Sidney A. Morris, Graduate School of ITMS, University of Ballarat, P.O. 663, Ballarat, Victoria 3353, Australia, and School of Engineering and Mathematical Sciences, La Trobe University, Bundoora Victoria 3086, Australia
E-mail: morris.sidney@gmail.com