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A Topological Group Characterization of Those Locally Convex Spaces Having Their Weak Topology

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In this note we give a surprisingly simple characterization of those locally convex topological vector spaces which have the weak topology. An application to varieties of topological groups is then given.

Theorem. *Let E be a locally convex Hausdorff real topological vector space. Then E has its weak topology if and only if every discrete subgroup (of the additive group structure) of E is finitely generated.*

Proof. If E has its weak topology then, as is well known, it is a subspace of a product of copies of the reals. Therefore, by Lemma 2 of [4], every discrete subgroup of E is finitely generated.

Suppose E does not have its weak topology. If F denotes the vector space spanned by the unit vectors $\{e_n\}$ in the Banach space l_1 then, by Theorem 1.4 of [5], F is a continuous linear image of a subspace of E. Noting that the subgroup of F generated by the $\{e_n\}$ has the discrete topology, we see that E has a discrete subgroup which is not finitely generated. The proof is complete.

Remark. The above result is also true for complex vector spaces.

It is proved in [2] that no infinite-dimensional normed vector space is in the variety of topological groups [3] generated by the class of all locally compact groups. Our corollary, below, says somewhat more.

Corollary. *Let $\mathscr{C}$ be the class of all locally compact groups. The locally convex Hausdorff real topological vector space E, regarded as a topological group, is in the variety generated by $\mathscr{C}$ if and only if E has its weak topology.*

Proof. By Theorem 4 of [1], if E is in the variety generated by $\mathscr{C}$ then it is a subgroup of a product $\prod_{i \in I} F_i$, where each F_i is a locally compact Hausdorff group. Let G_i be the closure in F_i of the projection of E. Clearly each G_i is a connected locally compact Hausdorff abelian group, and hence is compactly generated. Therefore, by the first corollary in [4]. $\prod_{i \in I} G_i$, and hence E, does not contain any discrete groups which are not finitely generated. Thus, by the above theorems, E must have its weak topology.

Conversely if E has its weak topology then it is a subspace of a product of copies of the reals and, consequently, is in the variety generated by $\mathscr{C}$.

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References

1. Brooks, M. S., Morris, Sidney A., Saxon, Stephen A.: Generating varieties of topological groups (submitted for publication).
2. Diestel, J., Morris, Sidney A., Saxon, Stephen A.: Varieties of linear topological spaces (in preparation).
3. Morris, Sidney A.: Varieties of topological groups. Bull. Austral. Math. Soc. **1**, 145—160 (1969).
4. — Locally compact abelian groups and the variety of topological groups generated by the reals. Proc. Amer. Math. Soc. (to appear).
5. Saxon, Stephen A.: Nuclear and product spaces, Baire-like spaces, and the strongest locally convex topology. Math. Ann. (to appear).

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