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A structure theorem on compact groups

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Introduction

We prove a new structure theorem which we call the Countable Layer Theorem. It says that for any compact group G we can construct a countable descending sequence $G = \Omega_0(G) \supseteq \cdots \supseteq \Omega_n(G) \cdots$ of closed characteristic subgroups of G with two important properties, namely, that their intersection $\bigcap_{n=1}^{\infty} \Omega_n(G)$ is $Z_0(G_0)$, the identity component of the center of the identity component G_0 of G, and that each quotient group $\Omega_{n-1}(G)/\Omega_n(G)$, is a cartesian product of compact simple groups (that is, compact groups having no normal subgroups other than the singleton and the whole group).

In the special case that G is totally disconnected (that is, profinite) the intersection of the sequence is trivial. Thus, even in the case that G is profinite, our theorem sharpens a theorem of Varopoulos [**8**], who showed in 1964 that each profinite group contains a descending sequence of closed subgroups, each normal in the preceding one, such that each quotient group is a product of finite simple groups. Our construction is functorial in a sense we will make clear in Section 1.

Let $\Lambda_n(G)$ denote the quotient group $\Omega_{n-1}(G)/\Omega_n(G)$ and remember that this is a product of finite simple groups if G is profinite. Included among the applications is the Topological Decomposition Theorem saying that for an arbitrary compact group G, the two compact groups G and $G_0 \times \prod_{n=1}^{\infty} \Lambda_n(G/G_0)$ are homeomorphic. As G/G_0 is profinite, $\prod_{n=1}^{\infty} \Lambda_n(G/G_0)$ is a product of finite simple groups which, if infinite, is by purely point set topological arguments in turn homeomorphic to a *Cantor cube* (that is a space of the form $\mathbf{2}^X$ with a discrete 2-element space $\mathbf{2}$ and a suitable set X). So we obtain, as an easy corollary, that each totally disconnected compact group is homeomorphic to a Cantor cube (cf. [**5**, pp. 96ff]). The Topological Decomposition Theorem also implies the Kuz'minov Theorem (cf. [**2**, p. 93]) that every compact group is a dyadic space, that is, a continuous image of a Cantor cube.

Furthermore, the Topological Decomposition Theorem yields a very explicit calculation of the topological weight of a compact group in terms of G_0 and algebraic invariants of G/G_0.

1. *Countable characteristic sequences*

As a first step towards the Countable Layer Theorem we examine functorially sequences of characteristic subgroups of compact groups. The key feature we notice is that these potentially transfinite sequences are in fact countable.

A subgroup N of a compact group G is called *subnormal* if there is a finite sequence $N = N_1 \trianglelefteq N_2 \trianglelefteq \cdots \trianglelefteq N_m = G$ of closed subgroups, each normal in the next. We shall write $N \trianglelefteq\!\trianglelefteq G$ whenever N is subnormal in G. A morphism $f: G \to H$ between compact groups will be called *normal*, respectively, *subnormal*, if $f(G) \trianglelefteq H$, respectively $f(G) \trianglelefteq\!\trianglelefteq H$. In [**6**, 9·72], it is shown that a subnormal connected closed subgroup of a compact connected group is normal and that a subnormal morphism between compact connected groups is normal. The composition of two normal morphisms is subnormal, the composition of two subnormal morphisms is subnormal.

We shall make reference to the category $\mathscr{C}$ of compact groups and all continuous morphisms between them and two subcategories, neither of them full:

$$\begin{aligned} \mathscr{C} &= \text{ category of all compact groups and all morphisms,} \\ \mathscr{CS} &= \text{ category of all compact groups and all surjective morphisms,} \\ \mathscr{CSN} &= \text{ category of all compact groups and all subnormal morphisms.} \end{aligned}$$

A morphism $f: G \to H$ of compact groups is called an *epimorphism* if it is surjective. By [**6**, p. 702, EA3·17] this is perfectly in accordance with category theoretical terminology. We shall write $\mathrm{Epi}\,(G, H)$ for the set of epimorphisms from G to H, that is, the expressions $\mathrm{Epi}\,(G, H)$ and $\mathscr{CS}(G, H)$ denote one and the same thing.

A *subfunctor* $\Sigma: \mathscr{A} \to \mathscr{A}$ on any of the categories listed above is a functor which attaches to each compact group G a subgroup $\Sigma G \leqslant G$ in such a fashion that for each $\mathscr{A}$-morphism $f: G \to H$ we have $f(\Sigma G) \subseteq \Sigma H$ and $\Sigma f: \Sigma G \to \Sigma H$ is the restriction and corestriction of f.

Definition 1·1. A possibly transfinite sequence of subfunctors $\Xi_\alpha: \mathscr{CS} \to \mathscr{CS}$, which assigns to each compact group G a descending chain of closed subgroups, indexed by ordinals $\alpha = 0, 1, \ldots, \omega, \omega + 1, \ldots, \zeta = \zeta(G)$,

$$G = \Xi_0 G \geqslant \Xi_1 G \geqslant \cdots \geqslant \Xi_\zeta G = \Xi_{\zeta+1} G$$

will be called a *strongly characteristic sequence* if the following conditions are satisfied:

(i) $(\forall \alpha < \zeta) \quad \Xi_{\alpha+1} G \neq \Xi_\alpha G$.

(ii) $\Xi_\lambda G = \bigcap_{\alpha<\lambda} \Xi_\alpha G$ for all limit ordinals $\lambda \leqslant \zeta$.

Saying that Ξ_α is a subfunctor on $\mathscr{CS}$ means that for each surjective continuous homomorphism $f: G \to H$ we have $f(\Xi_\alpha G) \subseteq \Xi_\alpha H$ and that $\Xi_\alpha f: \Xi_\alpha G \to \Xi_\alpha H$ is the restriction and corestriction of f. Since Ξ_α is a functor of $\mathscr{CS}$ into itself, the morphism Ξ_α is surjective. Hence

$$f(\Xi_\alpha G) = \Xi_\alpha H.$$

We further note that, in particular, *all $\Xi_\alpha G$ are characteristic, hence normal in G.*

It is clear that for each G, the sequence $\Xi_\alpha G$, properly descending because of (i), will *have to* stabilize at some smallest ordinal $\zeta(G)$ for reasons of cardinality. We shall write

$$\Xi_\infty(G) \stackrel{\text{def}}{=} \Xi_{\zeta(G)} G.$$

In particular, each strongly characteristic sequence attaches to each compact group a characteristic closed subgroup $\Xi_\infty G$. For all ordinals α beyond $\zeta(G)$ we write $\Xi_\alpha G = \Xi_\infty G$.

We hasten to present some examples. For subsets X and Y of a topological group G we let $[X, Y]$ denote the smallest *closed* subgroup containing all commutators $xyx^{-1}y^{-1}$, $x \in X$, $y \in Y$.

Example 1·2. We define, by transfinite recursion,

$$\Xi_\alpha G = \begin{cases} [G, \Xi_\beta] & \text{if } \alpha = \beta + 1, \\ \bigcap_{\beta<\alpha} \Xi_\beta G & \text{if } \alpha \text{ is a limit ordinal.} \end{cases} \tag{1}$$

Then Ξ_α is a strongly characteristic sequence, called the *descending central sequence*.

Proof. We must show that for each surjective morphism $f: G \to H$ we have $f(\Xi_\alpha G) = \Xi_\alpha H$. For each subgroup K of G write $\Sigma_G K = [G, K]$. One sees readily that $f(\Sigma K) = [H, f(K)] = \Sigma_H f(K)$. It now suffices to follow the pattern of the proof of Example 1·3 below.

Indeed, $\Xi_\alpha G/\Xi_{\alpha+1} G$ is central in $G/\Xi_{\alpha+1} G$ for all $\alpha \leqslant \zeta$.

We say that a compact group G is *transfinitely nilpotent* if $\Xi_{\zeta(G)} G = \{1\}$. A compact connected group is transfinitely nilpotent if and only if it is abelian ([**6**, 9·4]).

Example 1·3. Assume that $\Sigma: \mathscr{CS} \to \mathscr{CS}$ is a subfunctor. Then we can define, by transfinite recursion, a sequence

$$\Xi_\alpha G = \begin{cases} \Sigma(\Xi_\beta G) & \text{if } \alpha = \beta + 1, \\ \bigcap_{\beta<\alpha} \Xi_\beta G & \text{if } \alpha \text{ is a limit ordinal.} \end{cases} \tag{2}$$

Then $\Xi_\alpha G$ is a strongly characteristic sequence.

Proof. Let $f: G \to H$ be a surjective morphism. We prove by transfinite induction that $f(\Xi_\alpha G) = \Xi_\alpha H$. Since $\Xi_1 = \Sigma$ the assertion holds for $\alpha = 1$. Assume the assertion is true for all $\beta < \alpha$. If $\alpha = \beta + 1$, then $f(\Xi_\alpha G) = f(\Sigma(\Xi_\beta)) = \Sigma f(\Xi_\beta G)$ (since Σ is a subfunctor of $\mathscr{CS}$) $= \Sigma \Xi_\beta H$ (by the induction hypothesis) $= \Xi_\alpha H$ by the recursive definition of Ξ_α. Now assume that α is a limit ordinal. Then the intersection $\bigcap_{\beta<\alpha} \Xi_\beta G$ is also the limit of the filter basis of compact sets $\Xi_\beta G$, $\beta < \alpha$. Hence $f(\Xi_\alpha G) = f(\lim_{\beta<\alpha} \Xi_\beta G) = \lim_{\beta<\alpha} f(\Xi_\beta G)$ (by the continuity of f) $= \lim_{\beta<\alpha} \Xi_\beta H$ (by the induction hypothesis) $= \bigcap_{\beta<\alpha} \Xi_\beta H = \Xi_\alpha H$ again by the definition of Ξ_α.

The following special cases are of interest.

Example 1·4. Set $\Sigma G = [G, G]$. Then $\Sigma: \mathscr{C} \to \mathscr{C}$ is a subfunctor which induces a subfunctor on $\mathscr{CS}$. The sequence Ξ_α is strongly characteristic and $\Xi_\alpha G$ is called the *transfinite commutator sequence*.

A compact group is called *transfinitely solvable* if $\Xi_{\zeta(G)} G = \{1\}$. A compact connected group is transfinitely solvable if and only if it is abelian ([**6**, 9·4]). Every transfinitely nilpotent group is transfinitely solvable.

The next special case is of particular interest for us. By definition, a *compact simple group* is a compact topological group with precisely two closed normal subgroups. Examples are for instance SO(3), or A_5 (the 60 element simple group), or $\mathbb{Z}(2)$, $\mathbb{Z}(3)$

or isomorphic copies of these, at any rate. Note that while $SU(2)$, for example, is a simple Lie group in frequently used terminology, it is not simple in our sense.

Notation. Let $\mathscr{S}$ denote a set of representatives for the isomorphism classes on the class of all compact simple groups.

Then $\mathscr{S}$ is a countable set. (Indeed the set of isomorphism classes of connected compact simple groups is in bijective correspondence with the set of isomorphism classes of compact real Lie algebras which is in bijective correspondence with the set of isomorphism classes of complex simple Lie algebras which is in bijective correspondence with the set of isomorphism classes of Dynkin diagrams and that is countable (see e.g. [**1**]). The set of isomorphism classes of simple abelian groups is in bijective correspondence with the set of primes. The set of isomorphism classes of finite nonabelian simple groups is countable since, up to isomorphism, there is only a finite number of groups of order $n \in \mathbb{N}$. We shall assume that the abelian representatives *are* $\mathbb{Z}(p) \stackrel{\text{def}}{=} \mathbb{Z}/p\mathbb{Z}$, $p = 2, 3, 5, 7, \ldots$; when considered as fields, these are also written as $\mathrm{GF}(p)$.

Notation. Let G be a compact group.

(i) We write $\mathrm{Epi}\,(G, \mathscr{S})$ for $\dot{\bigcup}_{S\in\mathscr{S}} \mathrm{Epi}\,(G, S)$.

(ii) Set $\Omega(G) \stackrel{\text{def}}{=} \bigcap_{f\in\mathrm{Epi}\,(G,\mathscr{S})} \ker f$.

A simple group S has no subnormal subgroups other than $\{1\}$ and S. If $S \in \mathscr{S}$ then a subnormal morphism $f{:}\,G \to S$ is either the constant morphism const_{GS} or an epimorphism. Thus $\mathscr{C}\mathscr{S}\mathscr{N}(G, S) = \{\mathrm{const}_{GS}\} \cup \mathrm{Epi}\,(G, S)$ and $\Omega(G) = \bigcap_{f\in\mathscr{C}\mathscr{S}\mathscr{N}(G,S),\, S\in\mathscr{S}} \ker f$.

LEMMA 1·5. *$\Omega{:}\,\mathscr{C}\mathscr{S}\mathscr{N} \to \mathscr{C}\mathscr{S}\mathscr{N}$ is a subfunctor inducing a subfunctor $\Omega{:}\,\mathscr{C}\mathscr{S} \to \mathscr{C}\mathscr{S}$.*

Proof. Let $f{:}\,G \to H$ be a surjective $\mathscr{C}\mathscr{S}\mathscr{N}$-morphism. We have to show that $f(\Omega G) = \Omega H$. Since f is surjective, $f(f^{-1}(\Omega H)) = \Omega H$ and f induces a surjective morphism $G/f^{-1}(\Omega H) \to H/\Omega H$. It is no loss of generality to assume that $\Omega H = \{1\}$ and to show that $f(\Omega G) = \{1\}$, that is $\Omega H \subseteq \ker f$.

Let $S \in \mathscr{S}$ and let $\phi{:}\,H \to S$ be a morphism in $\mathrm{Epi}\,(H, S)$. Then $\phi\circ f \in \mathrm{Epi}\,(G, S)$, and thus $\Omega(G) \subseteq \ker \phi\circ f = f^{-1}(\ker \phi)$. Therefore

$$\Omega(G) \subseteq \bigcap_{\phi\in\mathrm{Epi}\,(H,\mathscr{S})} f^{-1}(\ker \phi) = f^{-1}\left(\bigcap_{\phi\in\mathrm{Epi}\,(H,\mathscr{S})} \ker \phi\right) = f^{-1}\Omega(H) = \ker f.$$

Now we are ready for another example of a strongly characteristic sequence.

Example 1·6. For any compact group G we define, by transfinite recursion,

$$\Omega_\alpha G = \begin{cases} \Omega(\Omega_\beta) & \text{if } \alpha = \beta + 1, \\ \bigcap_{\beta<\alpha} \Omega_\beta G & \text{if } \alpha \text{ is a limit ordinal.} \end{cases} \tag{3}$$

Then Ω_α is a strongly characteristic sequence; indeed every $\Omega_\alpha G$ is preserved under every subnormal endomorphism of G.

This sequence will be particularly valuable as we uncover the structure of each of the factor groups $\Omega_\alpha G/\Omega_{\alpha+1}G$ in the next section.

For the following we recall that compact Lie groups satisfy the descending chain condition for closed subgroups.

LEMMA 1·7. *Let G be a compact Lie group and $G_1 = G \geqslant G_2 \geqslant \cdots$ a descending sequence of closed subgroups. Then there is a natural number N such that $G_n = G_N$ for all $n \geqslant N$.*

Proof. We shall denote by $\mathscr{L}(H)$ the Lie algebra of a Lie group H. Now $\mathscr{L}(G_1) \geqslant \mathscr{L}(G_2) \geqslant \cdots$ is a descending sequence of finite dimensional vector spaces which becomes stationary from a certain natural number M on. Thus $(G_m)_0 = (G_M)_0$ for $m \geqslant M$. Let $f = \operatorname{card}(G_M/(G_M)_0)$ be the finite index of the identity component of G_M. Then there exist at most f natural numbers $n_j \geqslant M$ such that $G_{n_j} \neq G_{n_j+1}$. Let $N = 1 + \max_j n_j$.

COROLLARY 1·8. *Let Ξ_α be a strongly characteristic sequence of subfunctors on $\mathscr{C}\mathscr{S}$. Then for each compact* Lie *group G, the ordinal $\zeta(G)$ is finite.*

Proof. Since compact Lie groups satisfy the decending chain condition for closed subgroups by Lemma 1·7, the assertion is immediate.

The following theorem is important information on strongly characteristic sequences. Recall that ω denotes the first infinite ordinal.

THEOREM 1·9 (Countable Subfunctor Theorem). *Let Ξ_α be any strongly characteristic sequence of subfunctors on the category $\mathscr{C}\mathscr{S}$ of epimorphisms of compact groups. Then the cardinal $\zeta(G)$ at which the sequence for G becomes stationary is at most ω for each compact group G. That is, the descending sequence of characteristic closed subgroups $G = \Xi_0 G \geqslant \Xi_1 G \geqslant \Xi_2 G \geqslant \cdots$ becomes stationary after a finite number of steps or, at the latest, at the group $\Xi_\omega G = \bigcap_{n=0}^{\infty} \Xi_n G$.*

Proof. Let $\mathscr{N} \stackrel{\text{def}}{=} \{N \trianglelefteq G : G/N \text{ is a Lie group}\}$ and consider any $N \in \mathscr{N}$. Let $q: G \to G/N$ be the quotient map. Then

$$(\Xi_\infty G)N/N = q(\Xi_\infty G) = \Xi_\infty(G/N) = \Xi_\omega(G/N)$$

by Corollary 1·8. Hence

$$(\forall N \in \mathscr{N}) \quad (\Xi_\infty G)N = q^{-1}\Xi_\omega(G/N).$$

Furthermore, the equation $q(\Xi_\omega G) = \Xi_\omega(G/N)$ implies $q^{-1}\Xi_\omega(G/N) = \Xi_\omega(G)N$. Now $\bigcap_{N\in\mathscr{N}}(\Xi_\infty G)N = \Xi_\infty G$ and $\bigcap_{N\in\mathscr{N}} \Xi_\omega(G)N = \Xi_\omega G$ by [**6**, 9·1(ii)]. Therefore

$$\Xi_{\zeta(G)} = \Xi_\infty G = \Xi_\omega G.$$

We claim that this implies $\zeta(G) \leqslant \omega$; for if not, then $\omega < \zeta(G)$. Then by Definition 1·1(i) we would have

$$\Xi_{\zeta(G)} G = \Xi_\omega < \Xi_{\omega+1} G \leqslant \Xi_{\zeta(G)} G,$$

which would be a contradiction.

We turn to an immediate application. Let us say that a compact group is *countably nilpotent* if the intersection of the traditional descending central series

$$G, [G, G], \big[G, [G, G]\big], \ldots$$

is $\{1\}$, and let us say that a compact group is *countably solvable* if (it is solvable or) the intersection of the traditional commutator series

$$G, [G, G], \big[[G, G], [G, G]\big], \ldots$$

is $\{1\}$. Then we have at once:

Corollary 1·10. *A compact transfinitely nilpotent group is countably nilpotent and a compact transfinitely solvable group is countably solvable.*

Proof. This follows immediately from Theorem 1·9 and Examples 1·2 and 1·4.

Corollary 1·11. *The strongly characteristic sequence $G, \Omega G, \Omega(\Omega G), \ldots$, becomes stationary after a finite number of steps or at the latest with $\Omega_\infty G = \bigcap_{n=0}^{\infty} \Omega_n G$.*

Proof. This follows immediately from Theorem 1·9 and Example 1·6.

We shall investigate more thoroughly what Corollary 1·11 means in the next section. In particular, we will identify $\Omega_\infty G$.

For the concept of a projective limit, used in the following see [**6**, pp. 17ff] or [**9**, pp. 12ff].

Corollary 1·12. *In the notation of* Theorem 1·9,

$$G/\Sigma_\omega(G) = \lim_{\leftarrow} \big(G/\Sigma_1(G) \leftarrow G/\Sigma_2(G) \leftarrow G/\Sigma_3(G) \leftarrow \cdots\big). \tag{4}$$

Proof. By Theorem 1·9 we have $\bigcap_{n=1}^{\infty} \Sigma_n G/\Sigma_\omega G = \{1\}$ in the compact group $G/\Sigma_\omega G$. Then (4) follows from 1·33(ii) on p. 21 of [**6**].

It is noteworthy that for *any* strongly characteristic sequence Σ_n the compact group $G/\Sigma_\omega G$ is a projective limit of a *countable* inverse sequence of compact groups $G/\Sigma_n G$ which are technically simpler that $G/\Sigma_\omega G$ itself. We note, in this context, that there is no way to represent a nonmetrizable compact group as a projective limit of a *countable* inverse sequence of Lie groups.

We conclude this section by observing that the examination of descending sequences of subgroups (with operators) is a classical subject in the context of the Jordan–Hölder Theorem which, however, deals with sequences whose successive factor groups are simple. A good reference for finite groups is [**7**, pp. 55ff]. The first author to transport this formalism into the framework of profinite groups was Varopoulos [**8**, pp. 458–460]. Varopoulos' result has not received the attention it deserves.

2. *Strictly reductive compact groups*

In this section, we characterize those compact groups which have enough continuous homomorphisms onto compact simple groups to separate points as products of compact simple groups. This result will be called the Strict Reduction Theorem.

For our first definition in this section, recall that the countable set $\mathscr{S}$ is a fixed set of representatives for the class of compact simple groups.

Definition 2·1. For a compact group G and $S \in \mathscr{S}$, the smallest closed subgroup G_S of G containing all closed normal subgroups isomorphic to S is called the *S-socle* of G.

Clearly, G_S is a fully characteristic subgroup of G, that is, every (continuous!) endomorphism maps G_S into itself. Most often, G_S will be singleton, but in totally disconnected abelian groups, non-degenerate socles are common.

Definition 2·2. A compact group is called *strictly reductive* if it is (isomorphic to) a cartesian product of compact simple groups.

Proposition 2·3. *Assume that G is a strictly reductive compact group, and let $(G_S)_{S\in\mathscr{S}}$ denote the sequence of S-socles of G. Then there is a sequence of cardinals $\big(J(G,S)\big)_{S\in\mathscr{S}}$ such that*

$$G \cong \prod_{S\in\mathscr{S}} G_S, \quad G_S \cong S^{J(G,S)}.$$

Proof. Since G is a strictly reductive compact group we have $G \cong \prod_{k\in K} G_k$ for a family of compact simple groups G_k. For each $S \in \mathscr{S}$, set $K(S) \overset{\text{def}}{=} \{k \in K : G_k \cong S\}$ and define $J(G,S) = \operatorname{card} K(S)$. Then $K = \dot{\bigcup}_{S\in\mathscr{S}} K(S)$ and

$$G \cong \prod_{S\in\mathscr{S}} \prod_{k\in K(S)} G_k \cong \prod_{S\in\mathscr{S}} S^{J(G,S)}. \tag{5}$$

Let us write $G = \prod_{S\in\mathscr{S}} S^{J(G,S)}$. For $T \in \mathscr{S}$ set

$$P_T \overset{\text{def}}{=} \{(g_S)_{S\in\mathscr{S}} \in G : g_S = 1 \quad \text{if } S \neq T\} \cong T^{J(G,T)}.$$

Every factor $T_\alpha \overset{\text{def}}{=} \{(g_\beta)_{\beta\in J(G,T)} : \beta \neq \alpha \Rightarrow g_\beta = 1\}$ is a normal subgroup of $T^{J(H(G,T)}$, isomorphic to T, and corresponds to a subgroup of P_S which is normal in G and isomorphic to T. Since P_T is the smallest closed subgroup of G containing all these subgroups, we have $P_T \subseteq G_T$. Now let N be a closed normal subgroup of G isomorphic to T. The projection of N onto any simple factor of the product G is degenerate unless this factor is isomorphic to T. The common kernel of all projections $G \to S^{J(G,S)}$ with $S \neq T$ is exactly P_T. Hence $N \subseteq P_T$. Therefore $G_T \subseteq P_T$ by the definition of the T-socle G_T (see Definition 1·1). Thus $G_T = P_T$ and the Proposition is proved.

Remark. We shall, as a rule, consider the cardinals $J(G,S)$ as sets; this is often done anyhow when cardinals emerge as certain ordinals and ordinals are considered as sets (of predecessors). To call the sets $J(G,S)$ cardinals is just a device for expressing the fact, that the family of these cardinals determines G up to isomorphism.

Lemma 2·4. *Let $S \in \mathscr{S}$ and assume that on the compact group G the set of epimorphisms* $\operatorname{Epi}(G,S)$ *separates the points of G. Then $G \cong S^J$ for some set J.*

Proof. Let $\mathscr{N}$ be the set of all kernels of non-degenerate epimorphisms $f : G \to S$. Then $\bigcap \mathscr{N} = \{1\}$ by assumption.

(*a*) Assume that G is abelian. Then $S = \mathbb{Z}(p)$ for some prime p. Hence $G/N \cong \mathbb{Z}(p)$ for all $N \in \mathscr{N}$. Thus $N^\perp$, the annihilator of N in the character group $\hat{G}$, is isomorphic to $\mathbb{Z}(p)$. Then $\hat{G} = \{0\}^\perp = (\bigcap \mathscr{N})^\perp = \sum_{N\in\mathscr{N}} N^\perp$ is a $\mathrm{GF}(p)$ vector space and thus is isomorphic to $\mathrm{GF}(p)^{(J)}$ for some set J with card $J = \dim_{\mathrm{GF}(p)} \hat{G}$. Hence $G \cong \mathbb{Z}(p)^J$ by duality as asserted.

(b) Assume that G is non-abelian. Then S is a non-abelian compact simple group. We shall prove by induction that for every finite subset $\{N_1, \ldots, N_n\} \subseteq \mathscr{N}$ of n elements, the morphism

$$g \mapsto (gN_1, \ldots, gN_n): G \to G/N_1 \times \cdots \times G/N_n \qquad (6_n)$$

is surjective. If $n = 1$, there is nothing to prove. Assume that (6_n) holds for all n-element subsets of $\mathscr{N}$, and consider an $n+1$-element subset $\{N_1, \ldots, N_{n+1}\}$ of $\mathscr{N}$. Let D be the image of the morphism

$$g \mapsto (gN_1, \ldots, gN_{n+1}): G \to G/N_1 \times \cdots \times G/N_n \times G/N_{n+1}. \qquad (6_{n+1})$$

Then by (6_n) the projection of D onto the first n factors is surjective, and so is the projection on the last. For a proof of the claim that the morphism in (6_{n+1}) is surjective we may, after factoring $\bigcap_{j=1}^{n+1} N_j$ in all groups in sight, assume this intersection is trivial. Let $M \overset{\text{def}}{=} \bigcap_{j=1}^{n} N_j$, $P \overset{\text{def}}{=} MN_{n+1}$. Then $(m, n) \mapsto mn: M \times N_{n+1} \to P$ is an isomorphism and P is normal in G. Now G/P is a homomorphic image of both G/M and G/N_{n+1}. But G/N_{n+1} is simple. Thus either $P = G$ or $P = N_{n+1}$. In the first case we are finished since then $G = MN_{n+1}$ is a cartesian product with $M \cong S$ and $N_{n+1} \cong S^n$. The second case would mean $M \subseteq N_{n+1}$, and since $M \cap N_{n+1} = \{1\}$, this means $M = \{1\}$. Now we may write $G = S^n$, and since S is non-abelian simple, the normal subgroups are the partial products and there are exactly n normal subgroups N such that $G/N \cong S$. However, we assumed that there are $n+1$ of them. This contradiction rules out the second case and the induction is complete.

Now we define $\eta: G \to P \overset{\text{def}}{=} \prod_{N \in \mathscr{N}} G/N$ by $\eta(g) = (gN)_{N \in \mathscr{N}}$. Then η is injective and, since G is compact, it is an isomorphism onto its image. Let $\mathscr{M}$ be the filter basis of cofinite partial products M of P. By the surjectivity of the function (6_n) we see that $P = \eta(G)M$ for all $M \in \mathscr{M}$. Since $\mathscr{M}$ converges to 1 and $\eta(G)$ is closed by the compactness of G we conclude $\eta(G) = P$. Thus η is an isomorphism and the Lemma is proved.

Set $\boldsymbol{\Omega}_S(G) \overset{\text{def}}{=} \bigcap_{f \in \mathrm{Epi}(G,S)} \ker f$. The *S-cosocle* is the quotient group $\boldsymbol{\Gamma}_S(G) \overset{\text{def}}{=} G/\boldsymbol{\Omega}_S(G)$.

Lemma 2·5. *Assume that there are n different elements $S_1, \ldots, S_n$ in $\mathscr{S}$ such that the S-cosocle is nonsingleton exactly for $S \in \{S_1, \ldots, S_n\}$. Then $\boldsymbol{\Omega}(G) = \boldsymbol{\Omega}_{S_1}(G) \times \cdots \times \boldsymbol{\Omega}_{S_n}(G)$ and $\boldsymbol{\Gamma}(G) \cong \boldsymbol{\Gamma}_{S_1}(G) \times \cdots \times \boldsymbol{\Gamma}_{S_n}(G)$ via the map $g \mapsto \big(g\boldsymbol{\Omega}_{S_1}(G), \ldots, g\boldsymbol{\Omega}_{S_n}(G)\big)$.*

Proof. We prove the assertion by induction on n. For $n = 1$ the assertion is true trivially. Assume that it is true for n and let $S_1, \ldots, S_{n+1}$ be $n+1$ different elements of $\mathscr{S}$ such that $\boldsymbol{\Gamma}_S(G)$ is nontrivial exactly when S is one of these groups. Define

$$K \overset{\text{def}}{=} \boldsymbol{\Omega}_{S_1} \bigcap \cdots \bigcap \boldsymbol{\Omega}_{S_n}.$$

Then

$$G/K \cong \boldsymbol{\Gamma}_{S_1}(G) \times \cdots \times \boldsymbol{\Gamma}_{S_n}(G) \text{ via } gK \mapsto \big(g\boldsymbol{\Omega}_{S_1}(G), \ldots, g\boldsymbol{\Omega}_{S_n}(G)\big)$$

by the induction hypothesis (and the isomorphism theorem). Now $K \cap \boldsymbol{\Omega}_{S_{n+1}}(G) =$

$\Omega(G)$ by the assumption on the family of the S_j. Set $P = K\,.\,\Omega_{S_{n+1}}$. Then P is normal in G and $(k_1, k_2) \mapsto k_1 k_2 : K \times \Omega_{S_{n+1}} \to P$ is an isomorphism. The factor group G/P is a homomorphic image of both $G/K \cong \Gamma_{S_1}(G) \times \cdots \times \Gamma_{S_n}(G)$ and $\Gamma_{S_{n-1}}(G)$. By Lemma 1·4, $\Gamma_{S_{n-1}}(G) \cong S_{n+1}^{I_{n+1}}$ for some set I_{n+1}, and so $G/P \cong S_{n+1}^{I}$ for some $I \subseteq I_{n+1}$ (even if S_{n+1} is abelian). Similarly, $G/K \cong S_1^{I_1} \times \cdots \times S_n^{I_n}$ with all $I_m \neq \varnothing$. But now G/P can be a homomorphic image of this group only when $I = \varnothing$, that is $\Gamma(G) = P = K\,.\,\Omega_{S_{n+1}}$. Then $K \cong S^{I_{n+1}}(G)$ and $\Omega_{S_{n+1}} \cong \Gamma_{S_1}(G) \times \cdots \times \Gamma_{S_n}(G)$. Thus $I_{n+1} = J(G, S)$ and $G \cong \Gamma_{S_1}(G) \times \cdots \times \Gamma_{S_{n+1}}(G)$; this proves the claim of the Lemma.

THEOREM 2·6 (Strict Reduction Theorem). *For a compact group G, the following assertions are equivalent*:

(i) *G is strictly reductive*;

(ii) *the set* Epi $(G, \mathscr{S})$ *of all morphisms* $f: G \to S$, $S \in \mathscr{S}$ *separates the points of G, that is,* $\bigcap_{f \in \mathrm{Epi}\,(G,\mathscr{S})} \ker f = \{1\}$;

(iii) $\Omega(G) = \left[\bigcap_{f \in \mathrm{Epi}\,(G,\mathscr{S})} \ker f = \right]\{1\}$.

Proof. (ii) and (iii) are clearly equivalent.

(i)⇒(ii). Write $G = \prod_{S \in \mathscr{S}} S^{J(G,S)}$. Since all projections of G onto the simple factors belong to Epi $(G, \mathscr{S})$, and already these separate the points, the implication follows.

(ii)⇒(i). We reduce the problem. Set $P \overset{\mathrm{def}}{=} \prod_{S \in \mathscr{S}} \Gamma_S(G)$ and define $\eta: G \to P$ by $\eta(g) = (g\Omega_S(G))_{S \in \mathscr{S}}$. Then η is injective and, since G is compact, η is an isomorphism onto its image. Let $\mathscr{M}$ be the filter basis of cofinite partial products of P. Then Lemma 2·5 proves that for each $M \in \mathscr{M}$ we have $\eta(G)M = P$. Again since $\mathscr{M}$ converges to 1 and $\eta(G)$ is compact, hence closed, we conclude $\eta(G) = P$.

PROPOSITION 2·7. *Let G be a strictly reductive compact group and* $w(G)$ *its weight. Then*

$$w(G) = \begin{cases} \prod_{S \in \mathscr{S}} \mathrm{card}\,(S)^{J(G,S)} & \text{if } G \text{ is finite,} \\ \max\{\aleph_0, \sup_{S \in \mathscr{S}} J(G,S)\} & \text{if } G \text{ is infinite.} \end{cases} \tag{7}$$

Proof. We note that card $\mathscr{S} = \aleph_0$, $w(S) \leqslant \aleph_0$ for all $S \in \mathscr{S}$, and that for infinite G we can apply [**6**, p. 764, EA4·3] to our Example 1·3 to obtain $w(G) = \max\{\aleph_0, \sup_{S \in \mathscr{S}} w(S^{J(G,S)})\}$. Thus

$$w(G) = \max\{\aleph_0, \sup_{S \in \mathscr{S}} \max\{J(G,S), w(S)\} = \max\{\aleph_0, \sup_{S \in \mathscr{S}} J(G,S)\}.$$

3. *The Countable Layer Theorem*

With the machinery developed so far we now proceed to prove our main result, the Countable Layer Theorem.

COROLLARY 3·1. *Let* Ω *and* Ω_n *be as in Example* 1·6 *and* Corollary 1·11

$$\Omega_n G/\Omega_{n+1} G \textit{ is strictly reductive,}$$

that is, this quotient group is isomorphic to a cartesian product of simple compact groups for all $n < \zeta(G) \leqslant \omega$.

Proof. This is immediate from the definition of Ω and the Strict Reduction Theorem 2·7.

In the case of the descending central series and the descending commutator series of a compact group we obtain characteristic subgroups $\Xi_\infty G$; in the first case let us write $G^{[\infty]}$ and in the second case $G^{(\infty)}$. Then $[G, G^{[\infty]}] = G^{[\infty]}$, respectively, $[G^{(\infty)}, G^{(\infty)}] = G^{(\infty)}$. It is not clear how these characteristic subgroups should be characterized in other terms or identified as otherwise well known characteristic subgroups. In this regard, the situation is better for $\Omega_\infty G$.

For the following we recall the result in Corollary 1·11.

Lemma 3·2. *For a compact group G, the following statements are equivalent*:

(i) *G is connected and abelian*;

(ii) $G = \Omega(G)$.

Proof. (i)⇒(ii). If G is connected and abelian, then G has no homomorphic simple (non-degenerate) image.

¬(i)⇒¬(ii). If G is not connected then G/G_0 is nontrivially profinite and thus there is a nontrivial epimorphism $f: G \to S$ for some finite simple group S. Hence $\Omega G \subseteq \ker f \neq G$. If G is connected and not abelian then there is a nontrivial epimorphism $f: G \to S$ for some compact connected simple Lie group S. Again $\Omega G \subseteq \ker f \neq G$.

Proposition 3·3. *For each compact group, $\Omega_\infty G$ is $Z_0(G_0)$, the identity component of the centre of the identity component G_0 of G.*

Proof. If A is a compact connected abelian normal subgroup of G and $f \in \operatorname{Epi}(G, S)$ for some $S \in \mathscr{S}$, then $f(A)$ is a connected abelian closed normal subgroup of S and therefore $f(A) = \{1\}$. Hence $A \in \ker f$ and it follows that $A \subseteq \Omega G$. Recursively, it follows that $A \subseteq \Omega_n G$. Accordingly, $A \subseteq \Omega_\infty G$. This applies, in particular, to $A = Z_0(G_0)$.

Conversely, we claim that $Z_1 \stackrel{\text{def}}{=} \Omega_\infty G$ is abelian and connected. Suppose that it is not. Then by Lemma 3·2, $\Omega_{\zeta+1}(G) = \Omega(\Omega_\zeta(G)) \neq \Omega_\zeta(G) = Z_1$ which contradicts the definition of ζ. Thus the claim is proved. Now Z_1 is a connected compact abelian normal subgroup of G, hence of G_0. Since connected compact abelian normal subgroups of connected groups are central, we conclude $Z_1 \subseteq Z_0(G_0)$.

Summarizing the results regarding the strongly characteristic sequence Ω_n we obtain our main result.

Theorem 3·4 (Countable Layer Theorem). *Let G be an arbitrary compact group. Then the $\Omega_n G$ form a descending countable filtration, possibly stationary after a finite number of steps,*

$$G \geqslant \Omega G \geqslant \Omega_2 G \geqslant \cdots \geqslant \bigcap_{n=1}^{\infty} \Omega_n G = Z_0(G_0) \tag{8}$$

of characteristic closed subgroups of G, such that for each $n \in \mathbb{N} = \{1, 2, \ldots\}$ and each simple compact group $S \in \mathscr{S}$ there is a set $J_n(G, S)$ (representing a cardinal) such that

$$\Omega_{n-1} G / \Omega_n G \cong \prod_{S \in \mathscr{S}} S^{J_n(G,S)} \tag{9}$$

for all $n \in \mathbb{N}$. Moreover, the following propositions hold.

(i) *Among all descending filtrations*

$$G \geqslant N_1 \geqslant N_2 \geqslant \cdots$$

of closed subgroups N_m *which are normal in* G *and are such that* N_m/N_{m+1} *is strictly reductive for each* $m \geqslant 1$, *the sequence* (8) *descends fastest, that is,* $\Omega_n G = \Omega(\Omega_{n-1}G)$ *for all* $n \in \mathbb{N}$, *where* $\Omega(H)$ *is the common kernel of all morphisms of* H *onto some compact simple group.*

(ii) *For each* $n \in \mathbb{N}$, *the assignment* $H \mapsto \Omega_n H$ *is a functor from the category* $\mathscr{CSN}$ *of compact groups and morphisms with subnormal image into itself, that is, if* $f: H_1 \to H_2$ *is a morphism of compact groups and* $f(H_1) \trianglelefteq\!\trianglelefteq H_2$, *then* $f(\Omega_n H_1) \subseteq \Omega_n H_2$ *for all* n.

Corollary 1·12 now specializes immediately to

Corollary 3·5. *In the notation of* Theorem 3·4, *the group* $G/Z_0(G_0)$ *is the countable projective limit*

$$\lim_{\leftarrow} (G/\Omega_1(G) \leftarrow G/\Omega_2(G) \leftarrow G/\Omega_3(G) \leftarrow \cdots). \tag{10}$$

By the Countable Layer Theorem 3·4, the descending chacarteristic sequence becomes stationary at the compact connected group $A \stackrel{\text{def}}{=} Z_0(G_0)$. We remind the reader that such a group A possesses a closed totally disconnected subgroup D obtained with the aid of the Axiom of Choice such that A/D is a torus, i.e. a product of circle groups (see [**6**, p. 376, 8·15]). Theorem 3·4 applies to D so that we have the following complement to the Countable Layer Theorem:

Proposition 3·6. *Let* A *be a compact connected abelian group, such as, e.g.* $A = Z_0(G_0)$ *in* Theorem 3·4. *Then there is a countable sequence of closed subgroups*

$$A \geqslant D = \Omega_0(D) \geqslant \Omega_1(D) \geqslant \Omega_2(D) \geqslant \cdots \geqslant \bigcap_{n=1}^{\infty} \Omega_n(D) = \{1\}$$

such that A/D *is a torus and* $\Omega_{n-1}D/\Omega_n D$ *is a cartesian product of cyclic groups of prime order.*

Since D is not canonical in A, the countable sequence descending from A is not a characteristic sequence; but it shares some features with the sequence in Theorem 3·4. What is unique, however, is dim A/D, the 'number' of factors in the torus (see [**6**, p. 379, 8·20; p. 384, 8·23]).

4. *The Topological Decomposition Theorem*

In this section we use the Countable Layer Theorem to establish a result we shall call the Topological Decomposition Theorem and we illustrate its applicability by deducing nice standard results on the topology of compact groups.

Lemma 4·1. *Assume that we are given an inverse system of compact spaces*

$$X_1 \xleftarrow{p_1} X_2 \xleftarrow{p_2} X_3 \cdots$$

and that there are homeomorphisms $f_n: X_n \to X_{n-1} \times Y_n$, $X_0 = \{1\}$ *singleton and*

$Y_1 = X_1$, *such that the following diagram commutes for* $n = 1, 2, \ldots$:

$$\begin{array}{ccc} X_n & \xleftarrow{\ p_n\ } & X_{n+1} \\ {\scriptstyle \mathrm{id}_{X_n}}\big\downarrow & & \big\downarrow{\scriptstyle f_{n+1}} \\ X_n & \xleftarrow[\ pr_{X_n}\ ]{} & X_n \times Y_{n+1}. \end{array} \tag{11_n}$$

Set $X \stackrel{\text{def}}{=} \lim_n\{\cdots X_n \stackrel{p_n}{\leftarrow} X_{n+1} \cdots\}$ *and* $Y \stackrel{\text{def}}{=} \prod_n Y_n$.

Then X *and* Y *are homeomorphic.*

Proof. Let us recall that the product of a family $\{Z_j : j \in J\}$ is the set of all functions $\alpha: J \to U \stackrel{\text{def}}{=} \bigcup_{j\in J} Z_j$ such that $\alpha(j) \in X_j$ for all $j \in J$. For $I \subseteq J$, the function $\alpha \mapsto \alpha|I: \prod_{j\in J} Z_j \to \prod_{i\in I} Z_i$ will be denoted p_I. We write $\mathbb{N}_n \stackrel{\text{def}}{=} \{1, \ldots, n\}$.

Claim 1. For each n there is a homeomorphism $F_n: X_n \to \prod_{m\in\mathbb{N}_n} Y_n$ such that the following diagram commutes

$$\begin{array}{ccc} X_n & \xleftarrow{\ p_n\ } & X_{n+1} \\ {\scriptstyle F_n}\big\downarrow & & \big\downarrow{\scriptstyle F_{n+1}} \\ \prod\limits_{m\in\mathbb{N}_n} Y_m & \xleftarrow[\ p_{\mathbb{N}_n}\ ]{} & \prod\limits_{m\in\mathbb{N}_{n+1}} Y_m. \end{array} \tag{12_n}$$

We prove this claim by induction. Claim (12_1) is true by (11_1). Assume that (12_{n-1}) has been proved for $n \geqslant 2$. We must construct $F_n: X_{n-1} \times Y_n \to \prod_{m\in\mathbb{N}_n} Y_m$. By the induction hypothesis we have a homeomorphism $F_{n-1}: X_{n-1} \to X_{n-2} \times Y_{n-1}$ satisfying (12_{n-1}). We identify $\prod_{m\in\mathbb{N}_{n-1}} Y_m \times Y_n$ and $\prod_{m\in\mathbb{N}_n} Y_m$ and define F_n to be the composition

$$X_n \xrightarrow{\ f_n\ } X_{n-1} \times Y_n \xrightarrow{F_{n-1}\times \mathrm{id}_{Y_n}} \prod_{m\in\mathbb{N}_{n-1}} Y_m \times Y_n = \prod_{m\in\mathbb{N}_n} Y_m.$$

Then we have a commutative diagram

$$\begin{array}{ccc} X_{n-1} & \xleftarrow{\ p_n\ } & X_n \\ {\scriptstyle \mathrm{id}_{X_{n-1}}}\big\downarrow & (11_n) & \big\downarrow{\scriptstyle f_n} \\ X_{n-1} & \xleftarrow[\ \mathrm{pr}_{X_{n-1}}\ ]{} & X_{n-1} \times Y_n \\ {\scriptstyle F_{n-1}}\big\downarrow & & \big\downarrow{\scriptstyle F_{n-1}\times \mathrm{id}_{Y_n}} \\ \prod\limits_{m\in\mathbb{N}_{n-1}} Y_m & \xleftarrow[\ p_{\mathbb{N}_{n-1}}\ ]{} & \prod\limits_{m\in\mathbb{N}_n} Y_m \end{array}$$

after our identification. Since the vertical map on the right is F_n by definition, we have (12_n). This proves Claim 1.

Claim 2. We have a commutative diagram of inverse systems in which the rows are limit diagrams, where we abbreviate $P_n = \prod_{m\in\mathbb{N}_n} Y_m$, and where $F: X \to L$ is the induced morphism:

$$\begin{array}{ccccccccc} X_1 & \xleftarrow{\ p_1\ } & X_2 & \xleftarrow{\ p_2\ } & X_3 & \cdots & \longleftarrow & X \\ {\scriptstyle F_1}\big\downarrow & & {\scriptstyle F_2}\big\downarrow & & {\scriptstyle F_3}\big\downarrow & \cdots & & \big\downarrow{\scriptstyle F} \\ P_1 & \xleftarrow[\ p_{\mathbb{N}_1}\ ]{} & P_2 & \xleftarrow[\ p_{\mathbb{N}_2}\ ]{} & P_3 & \cdots & \longleftarrow & L. \end{array}$$

Since all F_n are homeomorphisms, F is also a homeomorphism.

Claim 3. $L = Y$: We may assume that

$$L = \left\{ ((y_m^{(n)})_{m\in\mathbb{N}_n})_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \prod_{m\in\mathbb{N}_n} Y_n : p_{\mathbb{N}_n}((y_m^{(n+1)})_{m\in\mathbb{N}_{n+1}}) = (y_m^{(n)})_{m\in\mathbb{N}_n} \right\}.$$

By induction it follows that $y_m^{(n)} = y_m^{(n+1)}$, $m \in \mathbb{N}_n$. Now let $((y_m^{(n)})_{m\in\mathbb{N}_n})_{n\in\mathbb{N}} \in L$ and $m \in \mathbb{N}$; then for all $n, n' \geqslant m$ we have $y_m^n = y_m^{n'}$. We set $y_m = y_m^{(m)}$ and thus obtain

$$((y_m^{(n)})_{m\in\mathbb{N}_n})_{n\in\mathbb{N}} = ((y_m)_{m\in\mathbb{N}_n})_{n\in\mathbb{N}}.$$

Thus the function

$$(y_m)_{m\in\mathbb{N}} \mapsto ((y_m)_{m\in\mathbb{N}_n})_{n\in\mathbb{N}} : Y \to L$$

is a homeomorphism.

This completes the proof of the Lemma.

Definition 4·2. For a compact group G and each $1 \leqslant n < \omega$ we recall $\mathbf{\Omega}_0(G) = G$ and define the n-th *layer* of G to be

$$\Lambda_n(G) = \mathbf{\Omega}_{n-1}(G)/\mathbf{\Omega}_n(G). \tag{13}$$

By definition of $\mathbf{\Omega}_n(G)$ we know that each layer is a strictly reductive compact group. We let $p_n: G/\mathbf{\Omega}_{n+1}(G) \to G/\mathbf{\Omega}_n(G)$ denote the quotient morphism.

Lemma 4·3. *Assume that G is totally disconnected. For each $n = 1, 2, \ldots$ there is a homeomorphism $f_n: G/\mathbf{\Omega}_n(G) \to G/\mathbf{\Omega}_{n-1}(G) \times \Lambda_n(G)$ such that*

$$\begin{array}{ccc} G/\mathbf{\Omega}_n & \xleftarrow{p_n} & G/\mathbf{\Omega}_{n+1} \\ {\scriptstyle \mathrm{id}_{G/\Omega_n(G)}}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ G/\mathbf{\Omega}_n(G) & \xleftarrow[pr_{G/\Omega_n(G)}]{} & \big(G/\mathbf{\Omega}_n(G)\big) \times \Lambda_{n+1}(G). \end{array} \tag{14_n}$$

Proof. Since G is totally disconnected, the subquotient $\Lambda_n(G) = \mathbf{\Omega}_{n-1}(G)/\mathbf{\Omega}_n(G)$ is totally disconnected. The Lemma therefore follows now from [**6**, p. 540, theorem 10·36(iii)(*c*)] with $\mathbf{\Omega}_{n-1}(G)$ in place of G and $\mathbf{\Omega}_n(G)$ in place of H, and where the action is multiplication on the left.

Theorem 4·4 (Topological Decomposition Theorem). *Let G be a compact group. Then the compact groups G and*

$$G_0 \times \prod_{n\in\mathbb{N}} \Lambda_n(G/G_0) \cong G_0 \times \prod_{\substack{n\in\mathbb{N} \\ S\in\mathscr{S}}} S^{J_n(G/G_0,S)} \tag{15}$$

are homeomorphic.

Proof. By [**6**, p. 541, corollary 10·37] the groups G and $G_0 \times G/G_0$ are homeomorphic. It suffices therefore to consider the case of a totally disconnected group G. For such a group, G and $\prod_{n\in\mathbb{N}} \Lambda_n(G)$ are homeomorphic by Lemmas 4·3 and 4·2.

Definition 4·5. A topological space is called a *Cantor cube* if it is homeomorphic to the space $\mathbf{2}^J$ for some infinite set J where $\mathbf{2} = \{0, 1\}$ is the discrete two element space.

Every Cantor cube is a topological group, namely, $\mathbb{Z}(2)^J$. Let us record the following:

Lemma 4·6. (*a*) *For a compact metric space* X *the following statements are equivalent*:

(i) X *is homeomorphic to* $\mathbf{2}^{\mathbb{N}}$;

(ii) X *is totally disconnected and has no isolated points.*

(*b*) *Every compact metric space is a continuous image of* $\mathbf{2}^{\mathbb{N}}$.

Proof. (*a*): Cf. [**4**, p. 370, 6·2·A(c)]. (*b*): Cf. [**4**, p. 291, 4·5·9(*b*)].

It follows that every infinite product of finite sets is homeomorphic to a Cantor cube. In fact we have:

Lemma 4·7. *If* $\{X_j : j \in J\}$ *is an infinite family of finite sets with* $\operatorname{card} X_j > 1$ *for all* j, *then there is a homeomorphism*

$$X \stackrel{\text{def}}{=} \prod_{j\in J} X_j \to \mathbf{2}^J = \mathbf{2}^{w(X)}.$$

In particular, every infinite totally disconnected strictly reductive group is a Cantor cube. Therefore, we obtain

Corollary 4·8 [**5**, p. 95, theorem 9·15]. *Let* G *be a compact group. Either* G_0 *has finite index in* G *or* G *is homeomorphic to the product of* G_0 *and the Cantor cube* $\mathbf{2}^{w(G/G_0)}$.

We say that a group G acts on a space X *with stable isotropy* if all isotropy groups G_x are conjugate (cf. [**6**, p. 519, 10·5]).

Theorem 4·9. *Let* G *be a compact group acting with stable isotropy* G_x *on an infinite compact space* X. *Then* $w(X) = \max\{w(G/G_x), w(X/G)\}$.

Proof. Let H denote a closed subgroup of G to which all isotropy groups G_x are conjugate. Clearly, the right side is less than or equal to the left side. We have to show the reverse. If G/H is finite, then the orbit projection is covering map with finite fibre; the assertion is true in this clase and so we shall assume that G/H is infinite.

Let $\mathscr{U}$ be a a set of open subsets of G satisfying $UH = U$ such that $\{U/H : U \in \mathscr{U}\}$ is a basis of the topology of G/H of cardinality $w(G/H)$. The set of pairs $(U_1, U_2) \in \mathscr{U}^2$ such that $\overline{U_1} \subseteq U_2$ is still of cardinality $w(G/H)$. For each such pair $(U_1, U_2) \in \mathscr{P}$, intersection of all $\overline{U_1}N$, where N ranges through the set of all compact normal subgroups of G such that G/N is a Lie group is $\overline{U_1}$; hence there is a compact normal subgroup N_P of G such that G/N_P is a Lie group and $\overline{U_1}N_P \subseteq U_2$. If $g \in G$ and U^* is an open set of G such that $U^*H = U$ and $gH \in U^*$, then there is a pair $(U_1, U_2) \in \mathscr{U}^2$ with $\overline{U_1} \subseteq U_2$ such that $gH \subseteq U_1$ and $U_2 \subseteq U^*$. Thus there is a set $\mathscr{P}$ of pairs (U, N) such that

(i) N is a normal subgroup of G such that G/N is a Lie group;

(ii) $U \in \mathscr{U}$;

(iii) $\operatorname{card}(\mathscr{P}) = w(G/H)$, and that

(iv) $\{U/H : (\exists, N)\,(U, N) \in \mathscr{T}\}$ is a basis of the topology of G/H.

The set of all $(U_1 \cap \cdots \cap U_k, N_1 \cap \cdots \cap N_k)$ with $(U_1, N_1), \ldots, (U_k, N_k) \in \mathscr{P}$ satisfies (i), ... , (iv) as well, and we assume henceforth that $\mathscr{N} \stackrel{\text{def}}{=} \{N : (\exists U)\,(U, N) \in \mathscr{P}\}$ is a filter basis.

Now let $\mathscr{V}$ be a basis of the topology of X/G of cardinality $w(X/G)$.

Assume that $M \in \mathscr{N}$. Then the Lie group G/M acts with constant isotropy on X/M via $gM*(M.x) = M.(g.x)$ (cf. [**6**, p. 536, 10·31]). Notice that $G/HM \cong (G/M)/(HM/M)$ as homogeneous spaces. Let $p_M: X/M \to X/G \cong (X/M)/(G/M)$ be (essentially) the orbit map. By the Local Cross Section Theorem for Compact Lie Group Actions ([**6**, p. 538, 10·34]) there is a subfamily $\mathscr{V}_M \subseteq \mathscr{V}$ which is still a basis for the topology of X/G and is such that

$$(\forall V \in \mathscr{V}_M)\, p_M^{-1}(V) \cong \frac{G}{HM} \times V \tag{16}$$

under a G/M-equivariant homeomorphism (cf. [**6**, p. 521, 10·9]). Then for $V \in \mathscr{V}_M$,

$$\left\{\frac{UNM}{MH} \times (V \cap W) : (U, N) \in \mathscr{P} \quad \text{and} \quad W \in \mathscr{V}_M\right\} \tag{17}$$

is a basis for the topology of the G/M-space $G/MH \times V \cong p_M^{-1}(V)$ whose cardinality does not exceed $\max\{w(G/H), w(X/G)\}$. Let $\mathscr{B}_{V,M}$ denote its image in $p_M^{-1}(V)$. Since $\operatorname{card}(\mathscr{V}_M) \leqslant \operatorname{card}(X/G)$ the set $\mathscr{B}_M = \bigcup_{V \in \mathscr{V}_M} \mathscr{B}_{V,M}$ is a basis of the topology of X/M whose cardinality does not exceed $\max\{w(G/H), w(X/G)\}$. Let $q_M: X \to X/M$ denote the orbit map for the action of M and let $p_M^{-1}(\mathscr{B}_M) \stackrel{\text{def}}{=} \{p_M^{-1}(B) : B \in \mathscr{B}_M\}$. Since by (iii) above, $\operatorname{card}(\mathscr{N}) \leqslant w(G/H)$ the set

$$\mathscr{B} \stackrel{\text{def}}{=} \bigcup_{M \in \mathscr{N}} p_M^{-1}(\mathscr{B}_M)$$

is a basis of the topology of X whose cardinality does not exceed $\max\{w(G/H), w(X/G)\}$. Hence $w(X) \leqslant \max\{w(G/H), w(X/G)\}$ as asserted.

Corollary 4·10. *Assume that G is an infinite compact group. Then the following conclusions hold.*

(i) *Let $N \leqslant G$ be a closed not necessarily normal subgroup. Then*

$$w(G) = \max\{w(N), w(G/N)\}.$$

(ii) *Let H be a topological group such that $w(G) > w(H)$ and let $f: G \to H$ be a morphism of topological groups. Then $w(G/\ker f) < w(G)$ and $w(\ker f) = w(G)$.*

Proof. (i) Since N acts freely on the left and on the right of G, this is an immediate consequence of Theorem 4·9.

(ii) Since G is compact, so is $G/\ker f$ and thus this group is embedded into H. Therefore, $w(G/\ker f) \leqslant w(H) < w(G)$. By (i) above,

$$w(G) = \max\{w(\ker f), w(G/\ker f)\}.$$

Thus $w(\ker f) = w(G)$.

Proposition 4·10(i) is [**3**, lemma 6·1]. Theorem 4·9, however, is more general and appears to be of independent interest.

Let $\mathbb{I} = [0, 1]$ with the euclidean topology.

Corollary 4·11. *Let G be an infinite compact group. Then G contains a subspace homeomorphic to $\mathbb{I}^{\dim G} \times \mathbf{2}^{w(G/G_0)}$ and thus G contains a Cantor cube $\mathbf{2}^{w(G)}$.*

Proof. By [**6**, p. 484, line 6], the group G_0 contains a subset homeomorphic to $\mathbb{I}^{\dim G_0}$. By definition, $\dim G_0 = \dim G$ (cf. [**6**, p. 483, 9·53 and 9·54].) Thus Corollary 4·8 implies the first assertion. By Lemma 4·6, the standard Cantor set inside $\mathbb{I}$ is homeomorphic to $\mathbf{2}^{\aleph_0}$. Thus $\mathbb{I}^{\dim G}$ contains a set homeomorphic to $\mathbf{2}^{\aleph_0 \cdot \dim G} = \mathbf{2}^{w(G_0)}$ as $w(G_0) = \max\{\aleph_0, \dim G\}$ by [**6**, p. 607, 12·25]. Thus G contains a Cantor cube $\mathbf{2}^{w(G_0)} \times \mathbf{2}^{w(G/G_0)} \cong \mathbf{2}^{w(G)}$, since $w(G) = w(G_0) + w(G/G_0)$ by Corollary 4·10.

Corollary 4·12. *Every infinite compact group is dyadic, that is, is a continuous image of a Cantor cube. Specifically, if G is an infinite compact group, then it is continuous image of* $\mathbf{2}^{w(G)}$.

Proof. By [**6**, p. 499, 9·76], G_0 is a homomorphic image of a product $\mathbb{P}$ whose factors are isomorphic to $\hat{\mathbb{Q}}$ or are simple simply connected compact Lie groups, and $w(\mathbb{P}) = w(G_0)$. Every one of the factors is a continuous image of $\mathbf{2}^{\mathbb{N}}$ by Lemma 4·6(*b*). Thus G_0 is a continuous image of $\mathbf{2}^{w(G_0)}$. If G_0 has finite index, then $G_0 \times G/G_0$ is a continuous image of $\mathbf{2}^{w(G_0)} \times \mathbf{2}^{\mathbb{N}}$. If G/G_0 is infinite, then by Corollary 4·8, G is a continuous image of $\mathbf{2}^{w(G_0)} \times \mathbf{2}^{w(G/G_0)} = \mathbf{2}^{w(G)}$.

For the history of the above result see [**2**, p. 93, 3·6·1].

Lemma 4·13. *Let G be an infinite compact group. Then*

$$w(G) = \max\{\aleph_0,\ w(\Omega_\infty G),\ \sup_{n\in\mathbb{N}} w(\Omega_{n-1}G/\Omega_n G)\}. \tag{18}$$

Proof. Clearly, the right side is less than or equal to the left side. We have to show the reverse. By induction we conclude from Corollary 4·10(i) that

$$w(G/\Omega_n G) \leqslant \max\{\aleph_0,\ w(\Omega_{m-1}G/\Omega_m G); m = 1, 2, \ldots, n\}. \tag{19}$$

Now let $f_n: G/\Omega_\infty G \to G/\Omega_n G$ be the natural quotient map and let $\mathscr{V}_n$ denote a basis for the identity neighbourhoods of $G/\Omega_n G$ of cardinality $w\big(G/\Omega_n G\big)$. Then by Corollary 3·5 and [**6**, proposition 1·33, p. 22], the set

$$\{f_n^{-1}(U): U \in \mathscr{V}_n, \quad n \in \mathbb{N}\} \tag{20}$$

is a basis of the identity of $G/\Omega_\infty G$ and, by (19), the cardinality of this set does not exceed

$$\max\{\aleph_0, \sup_{n\in\mathbb{N}} w(\Omega_{n-1}G/\Omega_n G)\}. \tag{21}$$

Since $w(G) = \max\{w(G/\Omega_\infty G),\ w(\Omega_\infty G)\}$ by Corollary 4·10(i), the assertion of the Lemma follows.

Theorem 4·14 (Computation of the Weight of a Compact Group). *Let G be an infinite compact group. Let the cardinals $J_n(G, S)$ be as in the* Countable Layer Theorem 3·4. *Then*

$$w(G) = \max\{\aleph_0,\ w(Z_0(G_0)),\ \sup\{J_n(G, S): n \in \mathbb{N};\ S \in \mathscr{S}\}\} \tag{22}$$

$$= \max\{w(G_0), \sup\{J_n(G/G_0, S): n \in \mathbb{N};\ S \in \mathscr{S}, \quad S \textit{ finite}\}. \tag{23}$$

Proof. In Proposition 3·3 we showed that $\Omega_\infty G = Z_0(G_0)$. By Proposition 2·8 we have $\max\{\aleph_0, w(\Omega_{n-1}G/\Omega_n G)\} = \max\{\aleph_0, \sup_{S\in\mathscr{S}} J_n G, S)\}$. Therefore (19) implies

(22). The Topological Decomposition Theorem 4·4, gives us

$$w(G) = \max\left\{ w(G_0), w\left(\prod_{\substack{n\in\mathbb{N}\\ S\in\mathscr{S}}} S^{J_n(G/G_0,S)} \right) \right\}.$$

If the first cardinal is not smaller than the second, then (23) follows. If the second cardinal is bigger than the first, then it is infinite and equal to $\sup\{J_n(G/G_0,S) : n \in \mathbb{N};\ S\in\mathscr{S}, S \text{ finite}\}$ by Proposition 2·8. Therefore (23) follows again.

We conclude by establishing an upper bound for the generating degree of a compact group. For the details and the notation we refer to [**6**, chapter 10, pp. 596ff].

LEMMA 4·15. (i) *Let $S \in \mathscr{S}$ and let J be infinite. Then S^J has a suitable set of cardinality* card J *and* $s(S^J) = \text{card}\, J$.

(ii) *If G is strictly reductive, $G = \prod_{S\in\mathscr{S}} S^{J(G,S)}$, then $s(G) = \sum_{S\in\mathscr{S}} J(G,S)$.*

Proof. (i) Note $1 \leqslant s(S) < \omega$. The union in S^J of a suitable set if each factor is suitable, and its cardinal does not exceed card J. We note that a set of cardinality $<$ card J cannot topologically generate S^J. The proof of (ii) is analogous.

A slightly stronger assertion than the one appearing in Lemma 4·16 below is established in the proof of lemma 12·9 on p. 599 of [**6**].

LEMMA 4·16. *If G is a compact totally disconnected group, then*

$$s(G) \leqslant \sum_{n=1}^{\infty} s(\Lambda_n G).$$

PROPOSITION 4·17. *Let G be a compact group such that at least one of the cardinals $\sum_{S\in\mathscr{S}} J_n(G/G_0,S)$, $n = 1, 2, \ldots$, is infinite, then*

$$s(G) \leqslant \max\left\{ s(G_0),\ \sup_{n\in\mathbb{N}} \sum_{S\in\mathscr{S}} J_n(G/G_0,S) \right\}.$$

Proof. We note that for a closed normal subgroup N of a compact group G we have $s(G/N) \leqslant s(N) + s(G/N)$ (see [**6**, 12·20, p. 604]). Applying this with $N = G_0$ we get $s(G) \leqslant s(G_0) + s(G/G_0)$. From the Countable Layer Theorem 3·4 and from Lemma 4·16 above we get $s(G/G_0) \leqslant \sum_{n=1}^{\infty} s(\Lambda_n(G/G_0))$. Using Lemma 4·15 we get $s(\Lambda_n(G/G_0)) = \sum_{S\in\mathscr{S}} J_n(G,S)$ and since at least one of the sums $\sum_{S\in\mathscr{S}} J_m(G,S)$ is infinite, we have

$$\sum_{n=1}^{\infty} \sum_{S\in\mathscr{S}} J_n(G,S) = \sup_{n\in\mathbb{N}} \sum_{S\in\mathscr{S}} J_n(G,S).$$

Since this cardinal is infinite, we also get

$$s(G_0) + \sup_{m\in\mathbb{N}} \sum_{S\in\mathscr{S}} J_m(G,S) = \max\left\{ s(G_0), \sup_{m\in\mathbb{N}} \sum_{S\in\mathscr{S}} J_m(G,S) \right\}.$$

For an explicit formula of the generating degree of any compact *connected* group – such as for instance $s(G_0)$ – see [**6**, p. 605, theorem 12·22].

The Countable Layer Theorem is amenable, as we have seen, to proofs by induction. This newly observed phenomenon will be exploited elsewhere by the authors in the investigation of abelian subgroups of arbitrary compact groups.

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