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Subgroups of monothetic groups

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Dedicated to B. H. Neumann on his ninetieth birthday

Abstract. It is shown that every separable abelian topological group is isomorphic with a topological subgroup of a monothetic group (that is, a topological group with a single topological generator). In particular, every separable metrizable abelian group embeds into a metrizable monothetic group. More generally, we describe all topological groups that can be embedded into monothetic groups: they are exactly the abelian topological groups of weight $\leqslant \mathfrak{c}$ covered by countably many translations of every non-empty open subset.

1 Introduction

A recent result by the present authors [9] states that every separable topological group is isomorphic with a topological subgroup of a group topologically generated by two elements. This is a topological analogue of the Higman–Neumann–Neumann Theorem. It leads to a simple description of those topological groups which are embeddable into topological groups with two (equivalently, finitely many) generators: they are exactly the topological groups that are $\aleph_0$-bounded (that is, covered by countably many translations of each non-empty open subset) and have weight $\leqslant \mathfrak{c}$.

Topological groups having one generator rather than two, that is, monothetic groups, are, naturally, abelian, and so are all their topological subgroups. Rather surprisingly, commutativity turns out to be the only restriction imposed on the previous result by reducing the number of generators from two to one.

Theorem 1.1. *Every separable abelian topological group embeds into a singly generated topological group.*

It is interesting to notice that, unlike the above mentioned topological version of the Higman–Neumann–Neumann theorem, our Theorem 1.1 has no apparent algebraic counterpart.

Corollary 1.2. *A topological group G is isomorphic with a topological subgroup of a monothetic group if and only if G is abelian, $\aleph_0$-bounded, and has weight* $\leqslant \mathfrak{c}$.

The present article was largely stimulated by the following, very general, question by Mycielski [11]: *what can be said about completely metrizable monothetic groups?* The following result gives some idea of the large size of such groups.

Theorem 1.3. *Every separable metrizable abelian group is isomorphic with a topological subgroup of a completely metrizable monothetic group.*

For example, all additive topological groups of separable Banach spaces are to be found among subgroups of complete metric monothetic groups.

2 Preliminary results and constructions

Let $X = (X, *)$ be a pointed set, that is, a set with a distinguished element $* \in X$. By $A(X, *)$ or simply $A(X)$ if no confusion can arise we shall denote the free abelian group on $X \backslash \{*\}$, having $*$ as its zero element, and by $L(X, *)$ a real vector space having $X \backslash \{*\}$ as its Hamel basis and $*$ as its zero vector. It is well known that $A(X, *)$ is canonically isomorphic with a subgroup of the additive group of $L(X, *)$, generated by X.

Let ρ be a pseudometric on X. As was shown by Graev [5], there exists a maximal translation-invariant pseudometric $\bar{\rho}$ on $A(X, *)$, whose restriction to X coincides with ρ. A similar result was established [1, 12] for the vector span of X: there exists a maximal prenorm p_ρ on $L(X, *)$, such that for every $x, y \in X$ one has $\rho(x, y) = p_\rho(x - y)$.

Since the pseudometric on $A(X)$ induced by the prenorm p_ρ is clearly translation invariant, for all $x, y \in A(X, *)$ one has $p_\rho(x - y) \leqslant \rho(x, y)$. The following important and non-trivial result, obtained by successive efforts of Tkachenko [14] and Uspenskiĭ ([15], pp. 660–662), shows that the two pseudometrics on $A(X)$ thus obtained in fact coincide.

Theorem 2.1 (Tkachenko–Uspenskiĭ). *If ρ is any pseudometric on a pointed set $(X, *)$, then $p_\rho(x - y) = \bar{\rho}(x, y)$ for all $x, y \in A(X, *)$.*

If now X is a Tychonoff topological space, then the *free abelian topological group* on X is the group $A(X)$ equipped with the finest group topology inducing the given topology on X as a subspace. Such a topology always exists, is Hausdorff, and has the universal property of the following kind: every continuous mapping f from X to an arbitrary abelian topological group G lifts to a unique continuous homomorphism $\bar{f} : A(X) \to G$. It was first observed by Graev that the topology of $A(X)$ is determined by the collection of all translation invariant pseudometrics of the form $\bar{\rho}$, where ρ is a continuous pseudometric on X. For an account of the theory of free abelian topological groups, see e.g. [8].

In a similar way, the *free locally convex space* on X is the vector space $L(X)$

equipped with the finest locally convex topology inducing the given topology on X. Such a topology exists and is Hausdorff whenever X is a Tychonoff topological space, and every continuous mapping f from X to an arbitrary locally convex space E extends to a unique continuous linear operator from $L(X)$ to E. The topology of $L(X)$ is determined by the collection of all prenorms of the type P_ρ; see [1, 12].

The following result was obtained by Graev [5].

Theorem 2.2 (Graev). *Let G be a Hausdorff topological group, and let $i : A(G) \to G$ be the unique continuous homomorphism from the free abelian topological group on (the underlying topological space of) G to the topological group G whose restriction to G is the identity map. Then i is a factor homomorphism of topological groups.*

The Tkachenko–Uspenskiĭ Theorem was put to use in [10], where the following technique was suggested. Let G be a topological group. Denote by $i : A(G) \to G$ the homomorphism described in Theorem 2.2, and let K denote the kernel of i. Then K is a closed topological subgroup of $A(G)$, and the topological factor group $A(G)/K$ is isomorphic to G. Moreover the same remains true if we consider the group $A(G)$ equipped with the Graev metric $\bar{d}$, where d is a translation-invariant metric generating the topology of G. According to the Tkachenko–Uspenskiĭ Theorem, $(A(G), \bar{d})$ is isomorphic to a topological subgroup of the normed space $(L(G), p_d)$. It is now easy to see that the topological factor group $L(G)/K$ contains G as a (closed) topological subgroup.

Proposition 2.3 ([10]). *Every metrizable abelian topological group is isomorphic to a topological subgroup of a topological factor group of the additive group of a suitable Banach space.*

We need to develop a slight technical variation on the above themes.

Let $\mathfrak{R}$ be a collection of pseudometrics on a pointed set $X = (X, *)$ inducing some (Tychonoff) topology. (In precise terms, the collection of all open balls formed with respect to pseudometrics from $\mathfrak{R}$ forms a topology base.) We will denote by $A(X, *, \mathfrak{R})$ the free abelian group $A(X, *)$, equipped with the collection of Graev pseudometrics $\overline{\mathfrak{R}} = \{\bar{\rho} : \rho \in \mathfrak{R}\}$. Mostly we shall be viewing $A(X, *, \mathfrak{R})$ as an abelian topological group under the (group) topology generated by all pseudometrics from $\overline{\mathfrak{R}}$. The space X evidently is a topological subspace of $A(X, *, \mathfrak{R})$. The following is also clear.

Proposition 2.4. *Let $(X, *, \mathfrak{R})$ be as above. Let G be an abelian topological group, and let $\mathfrak{P}$ be some collection of translation-invariant pseudometrics on G generating the topology. Let $f : X \to G$ be a mapping sending $*$ to 0_G and such that for every element $d \in \mathfrak{P}$ there is a $\rho \in \mathfrak{R}$ making the map $f : (X, \rho) \to (G, d)$ Lipschitz. Then the (unique) algebraic homomorphism $\bar{f} : A(X) \to G$, extending f, is continuous.*

In a similar way, we define a locally convex space $L(X, *, \mathfrak{R})$ as the linear span $L(X, *)$ of X with $*$ serving as zero equipped with the (locally convex topology generated by) the collection of prenorms $\tilde{\mathfrak{R}} = \{p_\rho : \rho \in \mathfrak{R}\}$. The space X is a topological subspace of $L(X, *, \mathfrak{R})$. One has the following counterpart of Proposition 2.4.

Proposition 2.5. *Let E be a locally convex space, and let $\mathfrak{P}$ be some collection of prenorms generating the topology of E. Let $f : X \to E$ be a mapping sending $*$ to 0_E and such that for every $p \in \mathfrak{P}$ there is a $\rho \in \mathfrak{R}$ making the map $f : (X, \rho) \to (E, p)$ Lipschitz. Then the (unique) linear operator $\bar{f} : L(X, *, \mathfrak{R}) \to E$, extending f, is continuous.*

Now Theorem 2.1 implies the following.

Corollary 2.6. *The additive topological group of the locally convex space $L(X, *, \mathfrak{R})$ contains an isomorphic copy of the abelian topological group $A(X, *, \mathfrak{R})$ in a canonical way.*

If $X = (X, *)$ is a pointed Tychonoff topological space, we will denote by $\mathfrak{R}(X)$ the collection of all continuous pseudometrics on X. Then $A(X, *, \mathfrak{R}(X))$ is naturally isomorphic to the Graev free abelian topological group $A(X)$ on X, cf. [5, 8], while $L(X, *, \mathfrak{R}(X))$ is naturally isomorphic to the Graev free locally convex space $L(X)$ on X, cf. [1, 4, 12].

Let G be an abelian topological group, and let $\mathfrak{R}$ be an arbitrary family of continuous translation-invariant pseudometrics generating the topology of G. The identity mapping Id_G satisfies the assumption of Proposition 2.4 and therefore extends to a unique continuous homomorphism onto $i : A(X, *, \mathfrak{R}) \to G$, sending $*$ to 0.

Proposition 2.7. *The homomorphism $i : A(X, *, \mathfrak{R}) \to G$ is open.*

Proof. In fact, the same homomorphism i is open even if considered as a mapping from the free abelian topological group $A(G)$ to G; see [2].

Denote by K_G the kernel of i, which is a closed topological subgroup of $A(X, *, \mathfrak{R})$. The openness of i implies the following result.

Corollary 2.8. *G is canonically topologically isomorphic to the topological factor group $A(X, *, \mathfrak{R})/K_G$.*

Let G be a countable topological group. Now choose as $\mathfrak{R}$ the collection of all translation-invariant continuous pseudometrics on G which are bounded by 1. Then $\mathfrak{R}$ determines the topology of G (which is true of every topological group G, cf. [5]). Denote by d the discrete metric on G, that is, one taking values 0 and 1 only. (In general, d is discontinuous, unless of course G is discrete.) Notice that for each $\rho \in \mathfrak{R}$ the identity mapping $(G, d) \to (G, \rho)$ is Lipschitz-1. This implies that the identity isomorphism $(A(G), \bar{d}) \to (A(G), \mathfrak{R})$ is continuous. Noticing that the locally convex space $L(G, d)$ is separable metrizable and contains $(A(X), \bar{d})$ as a topological subgroup (Corollary 2.6), one arrives at the following result.

Lemma 2.9. *Every countable topological group G is isomorphic to a topological factor group of a group $A(G, \mathfrak{R})$, contained as a topological subgroup in a separable locally convex space $L(G, \mathfrak{R})$, admitting a finer separable metrizable locally convex topology.*

3 Rolewicz's lemma

Let us introduce the following *ad hoc* definition.

Definition 3.1. An abelian topological group G is called an *ω-torus* if it is topologically generated by the union of countably infinitely many subgroups topologically isomorphic to the circle group $\mathbb{T} \cong U(1)$.

The following was essentially proved by Rolewicz [13]. Even though he did not state the result in its full generality, the proof is his. The construction forms a rich source of monothetic groups beyond the locally compact case (cf. e.g. [3] and references therein). Therefore we find it very useful to state Rolewicz's Lemma in its full generality, and we believe that such a generalization is of interest on its own and not just in connection with the subsequent application in this article. In the proof we will stick to the multiplicative notation as more convenient in this particular context.

Theorem 3.2. *Every completely metrizable ω-torus is monothetic.*

Proof. Let G be topologically generated by the union of a countable sequence of its subgroups $\mathbb{T}_i$, $i = 1, 2, \dots$, each of which is topologically isomorphic to the circle group $\mathbb{T}$. Fix a translation invariant metric ρ generating the topology on G. For $i = 1, 2, \dots$ choose recursively a number $n_i \in \mathbb{N}$ and an element $x_i \in \mathbb{T}_i$ satisfying the following properties.

1. $\rho(x_i, 0) < 2^{-i}$.
2. The first n_i powers of the product $x_1 x_2 \dots x_i$ form a 2^{-i}-net in the compact subgroup $\mathbb{T}_1 \mathbb{T}_2 \dots \mathbb{T}_i$ of G.
3. Whenever $j > i$, the first n_i powers of the element x_j are contained in the ρ-ball of zero having radius 2^{-j}.

As the base of recursion, choose any element $x_1 \in \mathbb{T}_1$ having infinite order and contained in the $1/2$-neighbourhood of zero formed with respect to the metric ρ. To perform the recursive step, recall the classical Kronecker Lemma: if $x_1, \dots, x_n$ are rationally independent real numbers, then the n-tuple $(x_1', \dots, x_n')$ made up of their images under the factor homomorphism $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ to the circle group $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ generates an everywhere dense subgroup in the n-torus $\mathbb{T}^n$. Now assume that $x_1, \dots, x_{i-1}$ and $n_1, \dots, n_{i-1}$ with the properties (1)–(3) have been chosen. If the closed subgroup A_i generated by $x_1 \dots x_{i-1}$ coincides with $T_1 \dots T_i$, we set $x_i := 0$. Otherwise, A_i is a proper closed subgroup of the group $T_1 \dots T_i$, and clearly the latter is isomorphic to the topological direct sum $A_i \times \mathbb{T}_i$. Moreover, the compact abelian Lie group A_i is itself isomorphic to a torus group $\mathbb{T}^j$ of a suitable rank $j \leqslant i - 1$, and the image of the topological generator $x_1 \dots x_{i-1}$ in $\mathbb{T}^j$ under such an isomorphism is a j-tuple, say $(z_1, \dots, z_j)$, of rationally independent elements of the circle group. Now choose $x_i \in \mathbb{T}_i$ to be an element that is rationally independent of the elements $z_1, \dots, z_j$ and such that all the powers $x_i, x_i^2, \dots, x_i^{n_{i-1}}$ are contained in the ρ-neighbourhood of zero having radius 2^{-i}. It follows from Kronecker's Lemma that the powers of $x_1 \dots x_i$ are everywhere dense in $\mathbb{T}_1 \dots \mathbb{T}_i$. Consequently it is possible to choose n_i sufficiently large that the first n_i powers of $x_1 \dots x_i$ form a 2^{-i}-net in $\mathbb{T}_1 \dots \mathbb{T}_i$. The step of the recursion is thus accomplished.

We claim that the element $x = \prod_{l=1}^{\infty} x_l$, which is clearly well-defined since the metric ρ on G is complete, is a topological generator for G. It is enough to demonstrate that for every number $k \in \mathbb{N}$ the closure of the cyclic group $\langle x \rangle$ contains $\mathbb{T}_k$. Let $k \in \mathbb{N}$ and let $z \in \mathbb{T}_k$. Now let $i \geqslant k$ be arbitrary. Since $z \in \mathbb{T}_1 \mathbb{T}_2 \dots \mathbb{T}_i$, condition (2) implies the existence of an $m \in \{0, 1, 2, \dots, n_i\}$ such that the mth power of $x' = x_1 x_2 \dots x_i$ is at a distance less than 2^{-i} from z. The mth power of the remainder of the infinite product, $x(x')^{-1} = \prod_{l=i+1}^{\infty} x_l$, is at a distance from zero which is less than

$$\sum_{l=i+1}^{\infty} \rho(0, (x_l)^m) < \sum_{l=i+1}^{\infty} 2^{-l} = 2^{-i}.$$

(Here we have used condition (3).) Finally,

$$\rho(x^m, z) \leqslant \rho((x')^m, z) + \rho((x(x')^{-1})^m, 0) < 2^{i-1}.$$

Since i can be chosen arbitrarily large, the latter inequality means that z is the limit of a sequence of suitable powers of x, and the proof is finished.

4 The main construction

Assume that we are given the following collection of data:

1. a separable topological vector space E;
2. a countable everywhere dense subset $X = \{x_m : m \in \mathbb{N}_+\}$ of E.

Form the direct sum topological vector space, equipped with the direct product topology:

$$H := E \oplus l_2(\mathbb{N}_+ \times \mathbb{N}_+).$$

We will identify E in a natural way with the topological vector subspace (and subgroup) first direct summand of H. Let $\{e_{m,n} : m, n \in \mathbb{N}_+\}$ be an orthonormal basis for $l_2(\mathbb{N}_+ \times \mathbb{N}_+)$ and define

$$\xi_{m,n} := (nx_m, e_{m,n}) \in H.$$

Let D denote the subgroup of H algebraically generated by $\{\xi_{m,n}; m, n \in \mathbb{N}_+\}$.

Let d be an arbitrary translation invariant continuous pseudometric on E. The family of all such pseudometrics generates the topology of E. Denote by $\tilde{d}$ the continuous translation invariant pseudometric on H defined by setting for each $x_1, x_2 \in E$ and $y_1,\ y_2 \in l_2(\mathbb{N}_+ \times \mathbb{N}_+)$

$$\tilde{d}((x_1, y_1), (x_2, y_2)) := d(x_1, x_2) + \|y_1 - y_2\|.$$

The collection of all pseudometrics of the form $\tilde{d}$ generates the topology of H.

Lemma 4.1. *The family of all neighbourhoods W of zero with the property*

$$(D + W) \cap E = W \cap E$$

is a neighbourhood basis in H.

Proof. Let d be an arbitrary pseudometric on E as above, and let

$$W := \{z \in H : \tilde{d}(z, 0) < 1\}.$$

To verify that W has the property in the statement of the lemma, it is clearly enough to show that for $z \in H$ and $x \in D$, $\tilde{d}(z + x, 0) \geqslant 1$ implies $\tilde{d}(z, 0) \geqslant 1$. Putting $y = z + x$, we see that it suffices to show that $\tilde{d}(x, y) \geqslant 1$.

An arbitrary element x of D is of the form

$$\sum_{i=1}^{s} k_i \xi_{m_i, n_i} \equiv \left(\sum_{i=1}^{s} k_i n_i x_{m_i}, \sum_{i=1}^{s} k_i e_{m_i, n_i} \right) \in H. \tag{4.1}$$

One can assume without loss in generality that all the integer coefficients k_i are non-zero, and that $(m_i, n_i) \neq (m_j, n_j)$ for $i \neq j$. Now assume that $x \neq 0$ and let $y \in E$ be arbitrary. (According to our earlier convention, we shall identify y with the element $(y, 0) \in H$.) One has

$$\begin{aligned} \tilde{d}(x, y) &= d\left(\sum_{i=1}^{s} k_i n_i x_{m_i}, y \right) + \left\| \sum_{i=1}^{s} k_i e_{m_i, n_i} \right\| \\ &\geqslant \left\| \sum_{i=1}^{s} k_i e_{m_i, n_i} \right\| \\ &\geqslant 1, \end{aligned} \tag{4.2}$$

and the claim follows.

Lemma 4.2. *The group D is discrete.*

Proof. Indeed, the image of D under the second coordinate projection $H \to l_2(\mathbb{N}_+ \times \mathbb{N}_+)$ (which is a homomorphism of topological vector spaces) is a discrete subgroup of $l_2(\mathbb{N}_+ \times \mathbb{N}_+)$, formed by all linear combinations of the standard basis elements with integer coefficients.

Lemma 4.3. *The linear span of* D *is everywhere dense in* H.

Proof. Let $m \in \mathbb{N}_+$ be arbitrary. For every $n \in \mathbb{N}_+$, the linear span of D contains the element $(1/n)\xi_{m,n} = (x_m, (1/n)e_{m,n})$, and for each continuous pseudometric d on E

$$\tilde{d}\left(\tfrac{1}{n}\xi_{m,n}, x_m\right) = d(x_m, x_m) + \left\| \tfrac{1}{n} e_{m,n} \right\| = \tfrac{1}{n}. \tag{4.3}$$

Consequently x_m is in the closed linear span of D. Since the set $\{x_m : m \in \mathbb{N}_+\}$ is everywhere dense in E, it follows that the closed linear span of D contains E. Further, each element of the form

$$e_{m,n} = \xi_{m,n} - \tfrac{1}{n} x_m,$$

where $m, n \in \mathbb{N}_+$, is in the closed linear span of D as well. But $\{e_{m,n} : m, n \in \mathbb{N}_+\}$ is an orthonormal basis for $l_2(\mathbb{N}_+ \times \mathbb{N}_+)$. The claim is established.

Lemma 4.4. *The factor group* H/D *is an* ω*-torus.*

Proof. According to Lemma 4.3, the topological group H is topologically generated by the union of countably many one-parameter subgroups passing through the elements of the form $\xi_{m,n}$. Therefore H/D is topologically generated by the union of images of all such one-parameter subgroups. But those images are tori, and there are countably many of them.

Lemma 4.5. *The restriction of the factor homomorphism* $\pi : H \to H/D$ *to* E *is a topological group isomorphism between E and its image in* H/D.

Proof. Since $D \cap E = \{0\}$, the homomorphism $\pi|_E$ is in fact an algebraic isomorphism, and clearly it is continuous. It remains to prove that $\pi|_E$ is open on its image. Let V be an arbitrary neighbourhood of zero in H. According to Lemma 4.1, there is a neighbourhood W of zero contained in V such that $(D + W) \cap E = W \cap E$, that is, $\pi(W \cap E) = \pi(W) \cap \pi(E)$. Consequently, the interior of $\pi(W \cap E)$ in $\pi(E)$ is nonempty (it contains the open set $\pi(W) \cap \pi(E)$ in the subspace topology induced from H/D), and the proof is finished.

Lemma 4.6. *If E is also complete and metrizable, then* H/D *is monothetic and metrizable.*

Proof. The metrizability of H, and therefore of H/D, is quite obvious, and monotheticity of H/D follows from Rolewicz's Lemma and Lemma 4.4.

Combining together Lemmas 4.6 and 4.5, we obtain the following.

Lemma 4.7. *Let E be a separable metrizable topological vector space. Then E embeds, as a topological group, into a monothetic metrizable group H.*

Our next task will be to obtain a similar result in the absence of metrizability.

Lemma 4.8. *Let the topological vector space E admit a finer Hausdorff topology $\mathfrak{T}$ that makes it into a separable metrizable topological vector space. Then the group H/D is monothetic.*

Proof. Denote by F the underlying vector space of E equipped with the topology $\mathfrak{T}$. The identity mapping $F \to E$ is a continuous linear operator, and as such, it extends over the completions of the two topological vector spaces in a unique way, giving rise to a continuous homomorphism (in general, no longer an algebraic isomorphism)

$$i : \hat{F} \to \hat{E}.$$

Choose a countable everywhere dense subset $X \subset F$, $X = \{x_m : m \in \mathbb{N}_+\}$. Then X remains everywhere dense in E as well. Now apply our construction to both spaces $\hat{E}$ and $\hat{F}$. To distinguish between the emerging pairs of objects, we will use subscripts E and F, respectively. Thus $H_F = \hat{F} \oplus l_2(\mathbb{N}_+ \times \mathbb{N}_+)$, etc. Obviously the subgroups D_E and D_F coincide as abstract groups, or, more precisely, $i|_{D_F} : D_F \to D_E$ is a topological group isomorphism. Consequently the homomorphism i factors through D_F to give rise to a continuous homomorphism $j : H_F/D_F \to H_E/D_E$. Evidently the image of j is an everywhere dense subgroup of H_E/D_E. Since the former of the two groups is completely metrizable, Rolewicz's Lemma coupled with Lemma 4.4 implies that H_F/D_F is a monothetic group. But the image of a monothetic group under a continuous homomorphism having dense image is again monothetic.

The following is a direct consequence of Lemmas 4.8 and 4.5.

Lemma 4.9. *Every topological vector space E that admits a finer Hausdorff topology $\mathfrak{T}$ making it into a separable metrizable topological vector space embeds as a topological subgroup into a monothetic group.*

5 Proofs of the main results

Denote provisionally by $\mathcal{G}$ the class of all topological groups that embed, as topological subgroups, into monothetic groups.

The following is obvious.

Lemma 5.1. *The class $\mathcal{G}$ is closed under passing to topological subgroups.*

The following is less evident.

Lemma 5.2. *The class $\mathcal{G}$ is closed under passing to topological factor groups.*

Proof. Indeed, let $G \in \mathcal{G}$, that is, for some monothetic group H, one has $G < H$. Let F be a closed subgroup of G. Denote by F' the closure of F in H; one has $F' \cap G = F$. Then it is a standard result (cf. e.g. [7]) that the topological factor group G/F is isomorphic to a topological subgroup of the factor group H/F' in a canonical way, and at the same time the group H/F' is clearly monothetic.

The proof of the next lemma is analogous to that of Corollary 1 of [9].

Lemma 5.3. *The class $\mathcal{G}$ is closed under passing to completions.*

Proof of Theorem 1.1. This follows from Lemmas 2.9, 5.1, 5.2, 5.3 and 4.9.

Proof of Corollary 1.2. It suffices to apply a description of topological subgroups of separable topological groups obtained in [9]: those are exactly the $\aleph_0$-bounded topological groups of weight $\leqslant \mathfrak{c}$. Both the statement and the proof remain true if we add the word 'abelian' throughout.

Proof of Theorem 1.3. Every separable metrizable abelian topological group G is isomorphic to a topological factor group of the free abelian group equipped with the Graev metric $A(G, \rho)$ where ρ is an arbitrary metric on G generating the topology. By the Tkachenko–Uspenskiĭ Theorem, $A(G, \rho)$ is isomorphic with a closed topological subgroup of the free Banach space $B(G, \rho)$. Notice that $B(G, \rho)$ is separable as well. According to Lemma 4.7, $B(G, \rho)$ is isomorphic with a topological subgroup of a suitable monothetic metrizable group. An application of Lemmas 5.1, 5.2, and 5.3 finishes the proof.

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