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LOCALLY COMPACT ABELIAN GROUPS AND THE VARIETY OF TOPOLOGICAL GROUPS GENERATED BY THE REALS

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ABSTRACT. An LCA group G can be "manufactured" from the group of reals, via repeated operations of taking quotients, subgroups and (arbitrary) cartesian products if and only if G is compactly generated.

1. **Introduction.** It is well known that every compactly generated LCA group can be "manufactured" from the group R of reals using (repeatedly) the operations of taking quotients, subgroups and cartesian products. It is shown here that these are the only LCA groups which can be so manufactured.

In the terminology of [**1**], [**2**], [**5**], [**6**] and [**7**] this means that an LCA group is in the variety generated by the reals if and only if it is compactly generated.

2. **Preliminaries.**

DEFINITION. A class V of topological groups is said to be a variety if it is closed under the operations of taking subgroups, quotient groups, arbitrary cartesian products and isomorphic images.

DEFINITION. Let G be a topological group and $V(G)$ be the intersection of all varieties containing G. Then $V(G)$ is said to be the variety generated by G.

NOTATION. If Ω is a class of topological groups, let (i) $S(\Omega)$, (ii) $Q(\Omega)$, (iii) $C(\Omega)$ and (iv) $D(\Omega)$ denote the class of all topological groups isomorphic to (i) subgroups of topological groups in Ω, (ii) quotient groups of topological groups in Ω, (iii) cartesian products of families of topological groups in Ω, and (iv) products of finite families of topological groups in Ω, respectively.

The following theorem is proved in [**1**].

THEOREM. *Let G be any abelian topological group and let E be a Hausdorff group in $V(G)$. Then $E \in SCQD\{G\}$.*

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3. **The theorem.** We begin with an alternative proof of a special case of Theorem 2.6 of [**8**]. (See Corollary 2 below and Corollary 3.22 of [**3**].)

LEMMA 1. *Let H be a compactly generated LCA group. If G is a discrete subgroup of H, then G is finitely generated.*

PROOF. Let Y be the subgroup of G consisting of all elements of finite order and X be the complement of Y. We will show that both Y and the group generated by X, $\mathrm{gp}\{X\}$, are finitely generated, and hence so too is G.

By Theorem 9.14 of [**4**], H is topologically isomorphic to $R^a \times Z^b \times F$, where Z is the discrete group of integers, F is a compact group and a and b are nonnegative integers. Let S be any finite subset of X. Then the group generated by S is Z^r, for some nonnegative integer r. By Theorem 9.12 of [**4**], $r \leqq a+b$. That is, any finitely generated subgroup of $\mathrm{gp}\{X\}$ is generated by $a+b$ elements. Thus $\mathrm{gp}\{X\}$ is finitely generated.

Let p_1, p_2 and p_3 be the natural projection mappings of H onto R^a, Z^b and F, respectively. Let y be any element in Y. Then, since y is of finite order, $p_1(y)=e_1$ and $p_2(y)=e_2$, where e_1 and e_2 denote identity elements. Thus $p_3(Y)$ is topologically isomorphic to Y. That is, $p_3(Y)$ is a discrete subgroup of F. Since F is compact, this implies $p_3(Y)$, and hence Y, is a finite set. The proof is complete.

LEMMA 2. *Let G be a discrete group in $V(R)$. Then G is finitely generated.*

PROOF. By the theorem in §2, $G \in SCQD\{R\}$. Using Theorem 9.11 of [**4**], we see, then, that G is a subgroup of a product $\prod_{i\in I} F_i$, where each F_i is topologically isomorphic to $R^{n_i} \times T^{m_i}$, T is the circle group and n_i and m_i are nonnegative integers.

Since G is discrete, there exist $\alpha_1, \cdots, \alpha_n$ in I such that $e=\prod_{i\in I} O_i \cap G$, where e is the identity element of G, each O_i is an open set in F_i, and for $i \neq \alpha_j$ for some $j \in \{1, \cdots, n\}$, $O_i = F_i$. Let $F=\prod_{j=1}^{n} F_{\alpha_j}$, and p be the natural projection mapping of $\prod_{i\in I} F_i$ onto F. Clearly, $p(G)$ is topologically isomorphic to G. Thus $p(G)$ is a discrete subgroup of F. However, by Lemma 1, this implies $p(G)$, and hence G is finitely generated.

THEOREM. *Let G be an LCA group in $V(R)$. Then G is compactly generated.*

PROOF. By Theorem 2.4.1 of [**9**], G contains an open subgroup H, where H is topologically isomorphic to $R^n \times F$, for some nonnegative integer n and compact group F.

Then the quotient group G/H is discrete and is in $V(R)$. Therefore, by Lemma 2, G/H is finitely generated. Thus G is an LCA group with a compactly generated subgroup H such that G/H is compactly generated. By 5.39 (i) of [**4**], this implies G is compactly generated.

COROLLARY 1. *Let H be a compactly generated LCA group and let G be an LCA group in $V(H)$. Then G is compactly generated.*

COROLLARY 2. *Let G be a subgroup of an arbitrary product of compactly generated LCA groups. If G is locally compact, then it is compactly generated.*

COROLLARY 3. *Let G be a subgroup of an arbitrary product of copies of R. If G is locally compact then it is topologically isomorphic to $R^a \times Z^b$, for some nonnegative integers a and b.*

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ERRATUM TO "LOCALLY COMPACT ABELIAN GROUPS AND THE VARIETY OF TOPOLOGICAL GROUPS GENERATED BY THE REALS"

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Professor U.N. Muhin has pointed out that the proof of Lemma 1 is wrong. "That is, any finitely generated subgroup of $gp\{X\}$ is generated by $a + b$ elements. Thus $gp\{X\}$ is finitely generated" is an incorrect deduction! As the lemma is a special case of Theorem 2.6 of [8], it is, of course, correct. It is possible to replace the proof we gave with a similar, but correct, one. However, a sneaky way to see that any closed subgroup of a compactly generated LCA-group is compactly generated is by observing that an LCA-group is compactly generated if and only if its dual group is a Lie group, and using the fact that any quotient of a Lie group is a Lie group.

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