ELSEVIER

Topology and its Applications 84 (1998) 105–120

TOPOLOGY AND ITS APPLICATIONS

Nondiscrete topological groups with many discrete subgroups

Sidney A. Morris [a,*], Viatcheslav N. Obraztsov [b,1]

[a] University of South Australia, City West Campus, North Terrace, Adelaide, SA 5000, Australia
[b] Department of Mathematics and Statistics, The University of Melbourne, Parkville, VIC 3052, Australia

Received 23 August 1995; revised 12 December 1995, 17 June 1996

Abstract

It is shown that for each positive integer n, there exists a nondiscrete Hausdorff topological group of cardinality $\aleph_n$ with no proper subgroup of the same cardinality and with each proper subgroup discrete. This result is typical of those proved here using a method introduced by A.Yu. Ol'shanskii. It is also shown that there exists a continuum of pairwise algebraically nonisomorphic nondiscrete Hausdorff topological groups, each of which contains every finite group of odd order and has all of its proper subgroups finite. © 1998 Published by Elsevier Science B.V.

Keywords: Group; Topological group; Hausdorff topology; Free amalgam; Cancellation diagram

AMS classification: 22A05; 20F05; 20F06

1. Free amalgams

Using the method of cancellation diagrams, Ol'shanskii and his (former) students have constructed many important examples of groups with prescribed properties (see, for example, [12]). On the other hand, it was shown by Shelah [14] and Ol'shanskii [11] that there are infinite nontopologizable groups, that is, groups which admit no Hausdorff topological group topology other than the discrete topology. In [7] we prove Theorem A below which is applied in this paper to produce several new results. Theorem A itself was proved by applying Ol'shanskii's method to the construction of nondiscrete Hausdorff topological groups with prescribed properties.

In order to formulate the results, we need the following definitions.

* Corresponding author. E-mail: Sid.Morris@unisa.edu.au.

[1] The author was supported by a grant from the Australian Research Council.

PII S0166-8641(97)00086-2

Definition 1. The *free amalgam* Ω^1 of an arbitrary set of groups $\{G_\mu\}_{\mu\in I}$ is defined to be the set

$$\bigcup_{\mu\in I} G_\mu$$

with $G_\mu \cap G_\nu = 1$ whenever $\mu \neq \nu$.

Definition 2. The mapping $g:\Omega^1 \to G$ is an *embedding* (respectively, *topological embedding*) of the free amalgam Ω^1 of a set of groups (respectively, topological groups) G_μ, $\mu \in I$, into a group (respectively, topological group) G if it is injective and G_μ is isomorphic (respectively, topologically isomorphic) to $g(G_\mu)$ for each $\mu \in I$.

Theorem A [7]. *The free amalgam Ω^1 of nontrivial groups $\{G_\mu\}_{\mu\in I}$, $|I| > 1$, without involutions can be embedded in a group $G = \mathrm{gp}\{\Omega^1\}$. Further, for each fixed cardinal number $\beta < |G|$ and normal subgroup L of G containing a nonidentity element in Ω^1, the group G admits a nondiscrete Hausdorff group topology such that*

(1) *G is a 0-dimensional group;*
(2) *every neighbourhood of the identity has cardinality $|G|$;*
(3) *if a subgroup H of G is conjugate to a subgroup of G_μ for some $\mu \in I$, or H is a cyclic subgroup, then H is discrete;*
(4) *every subgroup of G of cardinality $\gamma \leqslant \beta$ is discrete;*
(5) *L is an open subgroup of G.*

Remark 3. The assertions of Theorem A are true for each embedding of the free amalgam Ω^1 into G using the scheme of [12, §34] if the numbers $n_A, n_1, \ldots, n_h$ in the defining relations of rank i, $i \geqslant 1$, in [12, p. 271, (1) and (2)], are chosen to satisfy the following additional condition: $n_k \geqslant ni$ for each $k = 1, \ldots, h$ (for details see [12, pp. 270–272]).

Some words about the proof of Theorem A are in order. The topology on the resulting group G is introduced by constructing a complete system of neighbourhoods of the identity. These neighbourhoods are contained in L and consist of products of "long" and "short" words over the alphabet Ω^1, and every "long" word is l-aperiodic for small values of l. The study of such products is heavily based on Ol'shanskii's technique developed in [12].

In this paper we present some applications of Theorem A. But we would like to emphasize that this theorem allows one to reformulate many embedding results based on the scheme of [12, §34] (with the restriction that the resulting group G is not of finite exponent, being caused by the additional condition on the choice of the numbers $n_A, n_1, \ldots, n_k$, see Remark 3, and this helps to avoid examples of groups like the infinite nontopologizable group in [11], since the boundedness of the exponent plays an important role in the construction of that group) as a topological embedding of a set of discrete groups into a nondiscrete Hausdorff topological group.

2. Generating mappings

At present all embedding constructions, known to the authors, based on the scheme of [12, §34] are particular cases of an "economical" embedding scheme of a set of groups without involutions in a simple group with an arbitrary fixed outer automorphism group established in [10]. In this section we give the following simplified variant of the main result in [10].

Let $\{G_\mu\}_{\mu\in I}$ be an arbitrary set of nontrivial groups without involutions, Ω^1 the free amalgam of the groups G_μ, $\mu \in I$, and put $\Omega = \Omega^1 \backslash \{1\}$.

Definition 4. A mapping $f: 2^\Omega \backslash \{\emptyset\} \to 2^\Omega$ is called *generating* on the set Ω if the following conditions hold:

(1) if $C \subseteq G_\mu$ for some $\mu \in I$, then $f(C) = \mathrm{gp}\{C\} \backslash \{1\}$;

(2) if C is a finite subset of Ω and $C \not\subseteq G_\mu$ for each $\mu \in I$, then $f(C) = B$, where B is an arbitrary countable subset of Ω such that $C \subseteq B$ and if D is a finite subset of B, then $f(D) \subseteq B$;

(3) if C is an infinite subset of Ω, then $f(C) = \bigcup_{A\in T} f(A)$, where T is the set of all finite subsets of C.

For example, a generating mapping f on Ω can be defined in the following way: if $C \in 2^\Omega \backslash \{\emptyset\}$ and $C = \bigcup_{\mu\in I} C_\mu$, where $C_\mu = C \cap G_\mu$, $\mu \in I$, then $f(C) = (\bigcup_{\mu\in I} \mathrm{gp}\{C_\mu\}) \backslash \{1\}$ (we assume that $\mathrm{gp}\{C_\mu\} = \{1\}$ if $C_\mu = \emptyset$). It is obvious that in order to define a generating mapping f on Ω, it is sufficient to define it on finite subsets C of Ω such that $C \not\subseteq G_\mu$ for each $\mu \in I$.

The meaning of Definition 4, which played a central role in [10], is the following. If we would like to embed the free amalgam Ω^1 of the groups G_μ, $\mu \in I$, into a group G having a nontrivial normal subgroup L (possibly $G = L$) with a "well-described", but rather complicated if necessary, lattice of subgroups, then it is quite natural to construct G in such a way that every noncyclic subgroup of L is conjugate in G to a subgroup $\mathrm{gp}\{C\} \cap L$ for some nonempty $C \subseteq \Omega$. Now we note that a mapping $f: 2^\Omega \backslash \{\emptyset\} \to 2^\Omega$ defined by $f(C) = \mathrm{gp}\{C\} \cap \Omega$ is generating on the set Ω. One of the assertions of Theorem B given below is that for a given generating mapping f on Ω (with an additional unessential condition, see the statement of Theorem B), we can arrange an "economical" embedding (in the previous sense) of Ω^1 into $G = \mathrm{gp}\{\Omega\}$ with the property that $f(C) = \mathrm{gp}\{C\} \cap \Omega$ for each nonempty subset C of Ω. Thus we can define a subgroup lattice of the normal subgroup L of G by setting a generating mapping on Ω. For example, a Tarski monster of exponent p (that is, an infinite group all of whose proper subgroups are cyclic of prime order p) can be constructed (for a sufficiently large prime p) by embedding the free amalgam of a countable family of cyclic groups of order p into a group $G = L$ if we define $f(C) = \Omega$ for any $C \subseteq \Omega$ such that $C \not\subseteq G_\mu$ for each $\mu \in I$.

Let $G = \mathrm{gp}\{\Omega\}$ and f an arbitrary generating mapping on Ω. Every word over the alphabet Ω can be considered as an element of the group G, but this correspondence is not one-to-one, of course. We say that X is a *minimal word* in G if it follows from $X = Y$

in G that $|X| \leqslant |Y|$, where $|Z|$ denotes the length of the word Z. Let W be the set of all nonempty words over the alphabet Ω written in *normal form*; that is, every element X in W is written in the form $X_1 \cdots X_k$, where each X_l, $1 \leqslant l \leqslant k$, is a nontrivial element of $G_{\mu(l)}$, $\mu(l) \in I$, and $\mu(l) \neq \mu(l+1)$ for $l = 1, \ldots, k-1$. Then a mapping $F : 2^W \backslash \{\emptyset\} \to 2^\Omega$ is defined in the following way: if $C \subseteq W$ and $C \neq \emptyset$, then let V be the set of all letters occuring in the expressions of the words of C. Now set $F(C) = f(V)$.

A simplified version of the main result in [10] is

Theorem B [10]. *Let $g_\mu : G_\mu \to H$ be a set of arbitrary homomorphisms of groups with kernels N_μ, $\mu \in I$, such that a system of subgroups $\{g_\mu(G_\mu)\}_{\mu \in I}$ generates H. Let $\{N_\mu\}_{\mu \in I_1}$, $I_1 \subseteq I$, be the set of nontrivial groups of the set $\{N_\mu\}_{\mu \in I}$, and let Ω_1^1 be the free amalgam of the groups N_μ, $\mu \in I_1$. If $|I_1| > 1$ and f is an arbitrary generating mapping on Ω such that $f(C) \cap \Omega_1^1 \neq \emptyset$ if $C \not\subseteq G_\mu$ for each $\mu \in I$, then the free amalgam Ω^1 of the groups G_μ can be embedded in a group $G = \mathrm{gp}\{\Omega\}$ without involutions with the following properties*:

(1) *the free amalgam Ω_1^1 is embedded in a normal simple infinite subgroup L of G such that $G/L \cong H$*;
(2) $\mathrm{Aut}L \cong G$ *(and so* $\mathrm{Out}L \cong H$*) and if $g \in G_\mu \backslash \Omega_1^1$, $\mu \in I$, then the mapping $g : L \to g^{-1}Lg$ is a regular automorphism of L (that is, $g(a) = a$ if and only if $a = 1$) if and only if there is no $c \in G_\mu \cap \Omega_1$ such that $[g, c] = 1$, where $\Omega_1 = \Omega_1^1 \backslash \{1\}$. In fact, every automorphism of L is induced by an inner automorphism of G and $C_G(L) = \{1\}$*;
(3) *every subgroup M of G is either a cyclic group or $M \cap L = 1$ and the homomorphic image of M in $H \cong G/L$ has an element of infinite order, or M is noncyclic and conjugate in G to an extension $G_{C,H'}$ of a group H' by a normal subgroup $L_C = \mathrm{gp}\{C\} \cap L$ (that is, $G_{C,H'}/L_C \cong H'$), where $C \subseteq \Omega$ (we assume that $\mathrm{gp}\{C\} = 1$ if $C = \emptyset$) and $H' \leqslant H$. Further, if $C \not\subseteq G_\mu$ for each $\mu \in I$, then $G_{C,H'} \leqslant \mathrm{gp}\{C\}$*;
(4) *if X is a nontrivial minimal word in the group G and C is a nonempty subset of Ω, then $X \in \mathrm{gp}\{C\}$ if and only if $F(\{X\}) \subseteq f(C)$ (in particular, $X \in \Omega \cap \mathrm{gp}\{C\}$ if and only if $X \in f(C)$)*;
(5) *if $H = G_\nu$ for some $\nu \in I$ and the homomorphism $g_\mu : G_\mu \to H$ is trivial for each $\mu \in I \backslash \{\nu\}$, then G is the semidirect product of H and L*;
(6) *if all groups N_μ, $\mu \in I_1$, are periodic (respectively torsion-free), then G may be chosen so that the subgroup L is periodic (respectively torsion-free) too.*

Now we have all the necessary information for the derivation of some consequences from Theorem A.

3. Minimal topological groups

Our first applications of Theorem A will be devoted to *minimal* (*in the sense of Platonov*) topological groups, that is, nondiscrete Hausdorff topological groups all of whose proper closed subgroups are discrete. In 1965 Platonov posed a problem in [3, Problem 1.74] on the characterization of all such groups. Isiwata [2] and Robertson and Schreiber [13] proved that a locally compact abelian group G is minimal if and only if G is topologically isomorphic to $\mathbb{R}$, the additive group of real numbers with the usual topology, or $\mathbb{T}$ the compact circle group. Morris [5] extended this result by showing that the word "abelian" can be omitted, so that this characterization is valid in the class of locally compact groups.

We concentrate on constructions of strongly minimal topological groups.

Definition 5. A nondiscrete Hausdorff topological group G is called *strongly minimal* if every proper subgroup of G is discrete.

Of course, every strongly minimal topological group is minimal, and it follows from [5] that there are no locally compact strongly minimal topological groups. The simplest example of a strongly minimal topological group is a quasi-cyclic group with any nondiscrete Hausdorff topology. In this section we provide a large class of examples of strongly minimal topological groups.

First we prove a very simple result about strongly minimal topological groups.

Proposition 6. *If G is a strongly minimal topological group of cardinality α, then its local weight $\chi(G) \geqslant \alpha$, every neighbourhood of the identity 1 of G is of cardinality α, and there is no family $\{U_i\}_{1\leqslant i\leqslant\omega}$ of neighbourhoods of 1 such that $U_i = U_i^{-1}$ and $U_j^2 \subset U_i$ for each i, j, $1 \leqslant i < j \leqslant \omega$.*

Proof. Let U be a neighbourhood of the identity 1 of G and $|U| < \alpha$. Then $\mathrm{gp}\{U\}$ is a proper nondiscrete subgroup of G which is impossible.

If $\chi(G) < \alpha$, then there exists a local basis $\{W_i\}_{i\in L}$ for the topology of G with $|L| < \alpha$. Choosing one element in every set W_i, $i \in L$, we obtain a set M such that $|M| < \alpha$ and $\mathrm{gp}\{M\}$ is a proper nondiscrete subgroup of G.

Now we suppose that there is a family $\{U_i\}_{1\leqslant i\leqslant\omega}$ of neighbourhoods of 1 such that $U_i = U_i^{-1}$ and $U_j^2 \subset U_i$ for each i, j, $1 \leqslant i < j \leqslant \omega$. Then $M = \bigcap_{i<\omega} U_i$ is a proper open subgroup of G, since $U_\omega \subseteq M$, and we arrive at a contradiction to the definition of G. □

A source of countable strongly minimal topological groups is

Theorem C. *Let $\{G_\mu\}_{\mu\in I}$, $|I| > 1$, be a countable set of nontrivial countable discrete groups without involutions. Then the free amalgam Ω^1 of the groups G_μ can be topologically embedded in a countable simple strongly minimal topological group $G = \mathrm{gp}\{\Omega^1\}$ with the following properties:*

(1) *every proper subgroup of G is either a cyclic group or contained in a subgroup conjugate to some G_μ;*
(2) *if all groups G_μ, $\mu \in I$, are periodic (respectively torsion-free), then G may be chosen periodic (respectively torsion-free) too.*

Proof. Let H be the trivial group and $g_\mu : G_\mu \to H$ the trivial homomorphism for each $\mu \in I$. Then the system $\{N_\mu\}_{\mu\in I}$ of nontrivial kernels of the homomorphisms g_μ, $\mu \in I$, is the same as the set of groups G_μ, $\mu \in I$. We define a generating mapping f on $\Omega = \Omega^1 \backslash \{1\}$ in the following way: if $C \subseteq \Omega$ and $C \nsubseteq G_\mu$ for each $\mu \in I$, then $f(C) = \Omega$. Then Theorem B applies to Ω^1 and this mapping f and yields a simple group $G = \mathrm{gp}\{\Omega^1\}$, and properties (1) and (2) of the theorem follow from assertions (3), (4) and (6) of Theorem B and the definition of the mapping f.

Now Theorem A supplies the group G with a nondiscrete Hausdorff topology, and by Theorem A(3) and (1) of this theorem, the topological group G is a strongly minimal topological group. □

The proofs of Theorems F and G about uncountable strongly minimal topological groups are based on Theorem A and the following results from an unpublished manuscript [9] of the second author. The proofs of Theorems D and E were based in [9] on a generalization of an embedding scheme from [8]. Now these results can be deduced from Theorem B. Indeed, Theorem D in the case $n = 1$ is a simple corollary of Theorem B.

Theorem D [9]. *Let $\{G_\mu\}_{\mu\in I}$, $|I| > 1$, be an arbitrary set of nontrivial groups without involutions such that $\sum_{\mu\in I} |G_\mu| = \aleph_n$, for some positive integer n. Then the free amalgam Ω^1 of the groups G_μ can be embedded (using the scheme of* [12, §34]) *in a simple group $G = \mathrm{gp}\{\Omega^1\}$ such that if M is a proper subgroup of G and M is not contained in a subgroup conjugate in G to some G_μ, $\mu \in I$, then $|M| < \aleph_n$.*

Theorem E [9]. *If, for an infinite cardinal α, there exists a Jonsson group M of cardinality α (that is, M has no proper subgroups of cardinality α) without involutions, then M can be embedded (using the scheme of* [12, §34]) *in a simple Jonsson group G of cardinality α^+ without involutions.*

Proof of Theorem D for $n = 1$. Let H be the trivial group and all homomorphisms $g_\mu : G_\mu \to H$ are also trivial. We set $\Omega = \Omega^1 \backslash \{1\} = \{a_j \colon 1 \leqslant j < \omega_1\}$, where ω_1 is the first uncountable ordinal number. A generating mapping f on Ω is defined in the following way: if C is a finite subset of Ω such that $C \nsubseteq G_\mu$ for each $\mu \in I$, then let k be the maximal ordinal number such that a_k is contained in C, and we set

$$f(C) = \left(\bigcup_{\mu\in I} \mathrm{gp}\{\Omega(k) \cap G_\mu\} \right) \backslash \{1\},$$

where $\Omega(k) = \{a_j \colon 1 \leqslant j \leqslant k\}$ and $\mathrm{gp}\{\emptyset\} = \{1\}$. It is easy to see that this mapping f satisfies all conditions in Definition 4 of a generating mapping on Ω.

Theorem B applies to Ω^1 and f and yields a simple group $G = \mathrm{gp}\{\Omega^1\}$. Let M be a proper noncyclic subgroup of G. Then by Theorem B(3), M is conjugate in G to a subgroup generated by a nonempty subset C of Ω. If M is not contained in a subgroup conjugate in G to some group G_μ, $\mu \in I$, then $C \nsubseteq G_\mu$ for each $\mu \in I$. Hence the set C is countable, since otherwise it follows from the definition of the mapping f and

Theorem B(4) that $f(C) = \Omega$ and $M = G$, which completes the proof of Theorem D in the case $n = 1$. □

The proofs of Theorem D for $n > 1$ and Theorem E are more complicated, but they involve only Theorem B and some additional set-theoretic considerations. Since the manuscript [9] was deposited in VINITI in 1990 and so is not readily available outside the former Soviet Union, we have included the proofs of Theorems D and E in Section 5 to make this paper more self-contained.

The following result is an analogue of Theorem D for topological groups.

Theorem F. *Let $\{G_\mu\}_{\mu \in I}$, $|I| > 1$, be an arbitrary set of nontrivial discrete groups without involutions such that $\sum_{\mu \in I} |G_\mu| = \aleph_n$, for some positive integer n. Then the free amalgam Ω^1 of the groups G_μ can be topologically embedded in a simple strongly minimal topological group $G = \mathrm{gp}\{\Omega^1\}$ such that if M is a proper subgroup of G and M is not conjugate in G to a subgroup of any G_μ, then $|M| < \aleph_n$.*

Proof. The assertion of the theorem follows immediately from Theorem D and Theorem A with $\beta = \aleph_{n-1}$. □

Corollary 7. *For each positive integer $n \geqslant 1$, there exists a strongly minimal topological Jonsson group G of cardinality $\aleph_n$.*

Proof. It is sufficient to take a set $\{G_\mu\}_{\mu \in I}$, where $|I| = \aleph_n$ and G_μ to be the discrete infinite cyclic group for each $\mu \in I$, and apply Theorem F to the set $\{G_\mu\}_{\mu \in I}$. □

Corollary 7 leads us to pose an open question.

Open Question 1. For which cardinal numbers α, does there exist a strongly minimal (even minimal) topological group of cardinality α?

In [1] a problem was posed about the existence of Jonsson groups of cardinality $\aleph_\omega$. A positive answer to this question would give an opportunity to construct strongly minimal topological groups of "large" cardinalities.

Theorem G. *Assume that, for an infinite cardinal α, there exists a discrete Jonsson group M of cardinality α without involutions. Then M can be topologically embedded in a simple Jonsson strongly minimal topological group G of cardinality α^+ without involutions.*

Proof. By Theorem E, the group M can be embedded in a simple Jonsson group G of cardinality α^+ without involutions. Then it follows from Theorem A with $\beta = \alpha$ that G admits a nondiscrete Hausdorff topology such that G is a strongly minimal topological group. □

4. Further applications of Theorem A

The next application of Theorem A is devoted to the groups of outer topological automorphisms of nondiscrete groups and connected with a problem of Kargapolov [3, Problem 4.30] on the description of the automorphism groups of topological groups. Matumoto [4] proved that every group is algebraically isomorphic to the outer automorphism group of some group, and Theorem B presents an embedding scheme of an arbitrary set of groups without involutions in a simple infinite group with a "well-described" lattice of subgroups and a given outer automorphism group. Now we have

Theorem H. *Let $\{G_\mu\}_{\mu\in I}$, $|I| > 1$, be an arbitrary set of nontrivial discrete groups without involutions, H an arbitrary discrete group. Then there is a nondiscrete Hausdorff topological group G such that*

(1) *the free amalgam of the groups G_μ is topologically embedded in a simple normal infinite open subgroup L of G and $G/L \cong H$;*
(2) *the group $\mathrm{Aut}L$ of topological automorphisms of the group L is algebraically isomorphic to G (and the group $\mathrm{Out}L$ of outer topological automorphisms of L is algebraically isomorphic to H).*

Proof. Let $H = \mathrm{gp}\{h_j\}_{j\in M}$, and also let $\{S_j = \mathrm{gp}\{s_j\}\}_{j\in M}$ be a set of infinite cyclic groups. We define $g_\mu : G_\mu \to H$ to be the trivial homomorphism for each $\mu \in I$, and for each $j \in M$, we define a homomorphism $g_j : S_j \to H$ by setting $g_j(s_j^t) = h_j^t$, $t \geqslant 1$. Then Theorem B applies to the free amalgam Ω^1 of the groups $\{S_j\}_{j\in M}$ and $\{G_\mu\}_{\mu\in I}$, (and an arbitrary generating mapping f on $\Omega = \Omega^1\backslash\{1\}$ such that $f(C) \cap \Omega_1^1 \neq \emptyset$ if $C \not\subseteq S_j$ for each $j \in M$, where Ω_1^1 is the free amalgam of the kernels of the homomorphisms g_μ, $\mu \in I$, and g_j, $j \in M$) and yields a group G with a simple normal infinite subgroup L such that

(1) the free amalgam of the groups G_μ is embedded in L and $G/L \cong H$, and
(2) every automorphism of L is induced by an inner automorphism of G and $C_G(L) = \{1\}$.

Now it is time for Theorem A to be applied to the resulting group G. By Theorem A(5), we may assume that L is an open subgroup of G. It remains to note that every inner algebraic automorphism of a topological group is a homeomorphism. □

In [5] and [6] Morris records various characterizations, in the class of locally compact groups, of the additive group of real numbers $\mathbb{R}$ and the circle group $\mathbb{T}$. We show that there are many nondiscrete topological groups, other than subgroups of $\mathbb{R}$ and $\mathbb{T}$, satisfying these properties, and so these characterizations are not valid in the class of all nondiscrete Hausdorff groups.

In [6] the following characterizations of the group $\mathbb{T}$ were given:

(i) every proper closed subgroup has only a finite number of closed subgroups;
(ii) every proper closed subgroup is finite;
(iii) every proper closed subgroup is of the form $\{g\colon g^n = 1\}$, where n is any nonnegative integer and 1 denotes the identity element.

Now we have the following results.

Theorem I. *There exists a continuum of pairwise algebraically nonisomorphic quasi-finite (that is, countably infinite groups all of whose proper subgroups are finite) nondiscrete Hausdorff topological groups containing every finite group of odd order.*

Theorem J. *There exists a continuum of pairwise algebraically nonisomorphic nondiscrete Hausdorff topological groups in which all proper subgroups are finite cyclic and for each positive odd integer n, there is a cyclic subgroup of order n.*

It follows from the results of [5] that the group $\mathbb{R}$ can be characterized in the class of locally compact groups as a nondiscrete group all of whose proper closed subgroups are topologically isomorphic to $\mathbb{Z}$, where $\mathbb{Z}$ is the discrete group of integers.

Theorem K. *There exists a continuum of pairwise algebraically nonisomorphic nondiscrete Hausdorff topological groups all of whose proper subgroups are topologically isomorphic to $\mathbb{Z}$.*

Proof of Theorems I–K. The existence of the required groups follows from Theorem C. That there exists a continuum of the sets of pairwise algebraically nonisomorphic groups with the necessary properties can be proved in the same way as in the proof of Theorem 28.7 of [12]. The idea is that in the course of constructing a group G with the required properties, we use the defining relations of rank i, $i \in J$, in [12, p. 271, (2)] for an infinite set J of natural numbers. We also note that the numbers $n_1, \ldots, n_h$ (see Remark 3) can be chosen in such a way that if we replace, for some ranks, the last number n_h by n_h+2, then any of groups arising will share all the properties of G.

Now we choose an arbitrary subset K of J and construct a group G_K. The distinction between G_K and G is as follows. If $i \in K$, then we define the relations (2) of rank i for G_K in the same way as it was done for G, otherwise we replace the number n_h by n_h+2. It can be proved that, for $K \neq L$, the groups G_K and G_L are obtained as quotient groups of the two-generator free group F_2 by distinct normal subgroups (since otherwise both G_K and G_L must have involutions). Since the number of distinct homomorphisms of F_2 onto a fixed countable group G_K is countable, the set of all the groups G_K contains continuously many pairwise nonisomorphic groups, since there are continuously many distinct subsets K of the set J. □

It is easy to see that if a quasi-finite group G contains copies of the cyclic group of order 2^n for each $n \geqslant 1$, then G is the quisi-cyclic group C_{2^∞}, since an element $g \in G$ of order 2^n is in the centre $Z(G)$ of G (see, for example, [12, Corollary 7.4]) and so the group G is abelian. Therefore, there is no strongly minimal topological group G such that all proper subgroups of G are finite cyclic and every finite cyclic subgroup can be embedded in G. Thus it is quite natural to pose the following

Open Question 2. Let G be a nondiscrete Hausdorff topological group with the property that every proper closed subgroup of G is finite cyclic. If every finite cyclic group can be embedded in G, is G necessarily topologically isomorphic to a subgroup of the circle group $\mathbb{T}$?

All quasi-finite nondiscrete Hausdorff topological groups constructed in this paper are not of bounded exponent, because of Remark 3. On the other hand, we can produce the following modification of Ol'shanskii's example of an infinite nontopologizable group in [11].

Theorem L. *For any sufficiently large prime p, there exists a continuum of pairwise nonisomorphic infinite nontopologizable groups of exponent p^2 all of whose proper subgroups are cyclic. Every one of these groups is a central extension of a Tarski monster of exponent p by a cyclic central subgroup of order p.*

Proof. We repeat essentially the argument in the proof of Theorem 31.5 [12].

Let G be a Tarski monster of exponent p constructed using the scheme of [12, §25]. This construction induces the presentation $G = F/N$, where F is a free group. Next, we form an extension $T = F/[F,N]$ of G by the central subgroup $N' = N/[F,N]$. It follows from [12, Corollary 31.1] that N' is a free abelian group such that $N' = N_1' \oplus N_2'$, where N_1' is an abelian group freely generated by elements of the form $R_k = A_k^p[F,N]$, where A_k runs through the set of *periods* of all possible ranks. (The precise definition of periods of rank i, $i \geqslant 1$, can be found in [12, pp. 270, 271], but the reader can consider the set $\{A_k\}$ as a "minimal" set of elements of G with the property that every element of G is conjugate to a power of some period A_k.) Since all R_k are in the centre of $H = T/N_2'$, it follows that the subgroup L consisting of all products of the form $\prod_k R_k^{s_k}$, where $\sum_k s_k = 0$, is normal in H. It is obvious that the order of the coset $R_1L = R_2L = \cdots = C$ in the quotient group $A = H/L$ is infinite. Now we put $K = A/\mathrm{gp}\{C^p\}$. Thus, K is an extension of G by a cyclic central group $\mathrm{gp}\{C\}/\mathrm{gp}\{C^p\}$ which we call $\mathrm{gp}\{D\}$.

Now we verify that for any $X \in K \backslash \mathrm{gp}\{D\}$, we have that $X^p \in \mathrm{gp}\{D\} \backslash \{1\}$ in K. Since $K/\mathrm{gp}\{D\} = G$, we have that X is a conjugate in $K/\mathrm{gp}\{D\}$ of a power A_k^l of some period A_k, where $0 < l < p$. Then, replacing X by a conjugate, we may assume that $X = A_k^l D^t$ for some integer t. Hence

$$X^p = (A_k^l D^t)^p = (A_k^p)^l (D^p)^t = D^l \neq 1$$

in K, since $0 < l < p$. Thus, the set of all nontrivial elements of the group K is the union of finitely many sets of solutions to equations of the form $X^p = a$ (where the right-hand side a takes $p-1$ nontrivial values in $\mathrm{gp}\{D\}$) and the finite set $\mathrm{gp}\{D\} \backslash \{1\}$. Therefore, if K is given a Hausdorff topology, then the complement of the identity is closed in K, being the union of finitely many closed subsets, and the topology is discrete.

Since there are continuously many pairwise nonisomorphic Tarski monsters of exponent p constructed using the scheme of [12, §25] (see [12, Theorem 28.7] or the proofs of Theorems I–K), the proof of the Theorem is complete. □

Theorems I, J and L stimulate us into posing one more open question.

Open Question 3. Does there exist a quasi-finite group of bounded exponent which admits a nondiscrete Hausdorff topology?

5. Proofs of Theorems D and E

Let $\{G_\mu\}_{\mu\in I}$, $|I|>1$, be an arbitrary set of nontrivial groups without involutions, Ω^1 the free amalgam of the groups G_μ, $\mu\in I$, $\Omega=\Omega^1\backslash\{1\}$, and let α be the minimal ordinal number with $|W(\alpha)|=|\Omega|$, where $W(\alpha)=\{\beta\colon\ \beta<\alpha\}$. If $C\in 2^\Omega$, then $C=\bigcup_{\mu\in I}C_\mu$, where $C_\mu=C\cap G_\mu$, and we set $C'=(\bigcup_{\mu\in I}\mathrm{gp}\{C_\mu\})\backslash\{1\}$ (we assume that $\mathrm{gp}\{C_\mu\}=\{1\}$ if $C_\mu=\emptyset$).

Definition 8. A set $\{C_i\colon\ C_i\in 2^\Omega\backslash\{\emptyset\},\ i\in W(\alpha)\}$ is a *regular decomposition of the set Ω defined by a subset $\Phi=\{a_i\colon\ i\in W(\alpha)\}$ of Ω* if the following conditions hold:

(1) $\bigcup_{i<\alpha}C_i=\Omega$;

(2) $a_i\notin\bigcup_{j<i}C_j$ for each i, $1<i<\alpha$;

(3) $C_1=\mathrm{gp}\{a_1\}\backslash\{1\}$, and if $i>1$, then $C_i=(\{a_i\}\cup\bigcup_{j<i}C_j)'\backslash(\bigcup_{j<i}C_j)$.

Let $\alpha\geqslant\omega_1$. Then it is easy to see that Ω always has a regular decomposition (which is not unique). Indeed, Ω can be represented in the form $\Omega=\{b_j\colon\ j\in W(\alpha)\}$. We set $a_1=b_1$ and $C_1=\mathrm{gp}\{a_1\}\backslash\{1\}$. Now let $i\in W(\alpha)$, $i>1$, and assume that we have defined a_j and C_j for each $j<i$. We note that the set $\Omega\backslash(\bigcup_{j<i}C_j)\neq\emptyset$, since otherwise $|\Omega|<|W(\alpha)|$, and in this set we choose the element b_k with the minimal index k. Then we set $a_i=b_k$ and $C_i=(\{a_i\}\cup\bigcup_{j<i}C_j)'\backslash(\bigcup_{j<i}C_j)$. As a result, we obtain a regular decomposition $\{C_i\colon\ i\in W(\alpha)\}$ of Ω defined by the set $\Phi=\{a_i\colon\ i\in W(\alpha)\}$.

Lemma 9. *Assume that $\alpha\geqslant\omega_1$ and $\{C_i\colon\ C_i\in 2^\Omega\backslash\{\emptyset\},\ i\in W(\alpha)\}$ is a regular decomposition of Ω defined by a set $\Phi=\{a_i\colon\ i\in W(\alpha)\}$ such that $\{a_1,a_2\}\not\subseteq G_\mu$ for each $\mu\in I$ if $D=\bigcup_{i<\omega_1}C_i$ is not contained in any G_μ, $\mu\in I$. Then there is a generating mapping f on D such that*

(1) *$f(\bigcup_{j\leqslant i}C_j)=\bigcup_{j\leqslant i}C_j$ for each $i<\omega_1$;*

(2) *if $B\subseteq D$ and $B\not\subseteq G_\mu$ for each $\mu\in I$, then $a_1,a_2\in f(B)$;*

(3) *if $D\not\subseteq G_\mu$ for each $\mu\in I$, $b_1\in C_i$ and $b_2\in C_j$ for some $j<i<\omega_1$, then $f(\{b_1,b_2,a_1,a_2\})=f(\{a_i,a_1,a_2\})$;*

(4) *if $B\subseteq D$, $B\not\subseteq G_\mu$ for each $\mu\in I$ and $|B|=\aleph_1$, then $f(B)=D$.*

Proof. We define a mapping f on nonempty subsets of D in the following way. If $C\in 2^D\backslash\{\emptyset\}$ and $C\subseteq G_\mu$ for some $\mu\in I$, then we set $f(C)=\mathrm{gp}\{C\}\backslash\{1\}$, and if $C=\{b_1,\dots,b_k\}\subseteq D$, $k\geqslant 2$, $b_s\in C_{i_s}$, $1\leqslant s\leqslant k$, and $C\not\subseteq G_\mu$ for each $\mu\in I$, then we define $f(C)=\bigcup_{j\leqslant i}C_j$, where $i=\max(i_1,\dots,i_k)$.

It follows from Definition 8 of a regular decomposition of Ω that if $a_i\in\Phi$ and $a_i\in G_\mu$ for some $\mu\in I$, then $C_i\subseteq G_\mu$. Moreover, if we set $\Pi_{j,\nu}=\{a_t\colon\ t\leqslant j\}\cap G_\nu$, where $j\in W(\alpha)$ and $\nu\in I$, then $C_i\subseteq\mathrm{gp}\{\Pi_{i,\mu}\}$, and hence C_s is a countable set for each $s\in W(\omega_1)$. Therefore, if $C=\{b_1,\dots,b_k\}$ is a finite subset of D, then $f(C)$ is a countable set. It is obvious that $C \subseteq f(C)$. Now if $E = \{e_1, \ldots, e_p\}$ is contained in $f(C)$, then by the definitions of the mapping f and a regular decomposition of Ω, we obtain that $f(E) \subseteq f(C)$. Thus, completing the definition of f on infinite subsets of D (using Definition 4(3)), we obtain that f is a generating mapping on D.

Assertions (1)–(3) of the lemma follow immediately from the definition of the mapping f. Let $B \subseteq D$, $B \not\subseteq G_\mu$ for each $\mu \in I$ and $|B| = \aleph_1$. Then $B \not\subseteq \bigcup_{j \leqslant i} C_j$ for each $i \in W(\omega_1)$, since C_i is a countable set for each $i < \omega_1$. Hence by the definition of the mapping f, $f(B) = D$, which completes the proof of the lemma. □

Lemma 10. *Let the free amalgam Ω^1 of the groups G_μ, $\mu \in I$, be of cardinality α greater than $\aleph_1$, α_1 and α_2 the minimal ordinal numbers corresponding to cardinal numbers m and m^+, respectively, where $\aleph_1 \leqslant m < |\Omega|$. Let*

$$\{C_i \colon C_i \in 2^{\Omega}\backslash\{\emptyset\},\ i \in W(\alpha)\}$$

be a regular decomposition of Ω defined by a set $\Phi = \{a_i \colon i \in W(\alpha)\}$ such that $\{a_i, a_1, a_2\} \not\subseteq G_\mu$ for each $i \in W(\alpha_2)\backslash W(\alpha_1)$ and $\mu \in I$, and also let $D_1 = \bigcup_{i<\alpha_1} C_i$ and $D_2 = \bigcup_{i<\alpha_2} C_i$. If f_1 is a generating mapping on D_1, then there exists a generating mapping f_2 on D_2 with the following properties:

(1) *f_1 is the restriction of f_2 to D_1;*
(2) *$f_2(\bigcup_{j\leqslant i} C_j) = \bigcup_{j\leqslant i} C_j$, for each $i \in W(\alpha_2)\backslash W(\alpha_1)$;*
(3) *if $B \subseteq D_2$, $B \not\subseteq D_1$ and $B \not\subseteq G_\mu$ for each $\mu \in I$, then $a_1, a_2 \in f_2(B)$;*
(4) *if $b_1 \in C_i$ and $b_2 \in C_j$, where $\alpha_1 \leqslant i < \alpha_2$ and $j < i$, then there exists a finite subset E of D_1 such that $f_2(\{b_1, b_2, a_1, a_2\}) = f_2(\{a_i\} \cup E)$;*
(5) *assume that $f_1(C) = D_1$ for each subset C of D_1 such that $|C| = m$ and if the set $\{a_1, a_2\}$ is not contained in any group G_μ, $\mu \in I$, then $C \not\subseteq G_\mu$ for each $\mu \in I$.*

Then if $B \subseteq D_2$, $B \not\subseteq G_\mu$ for each $\mu \in I$ and $|B| = m^+$, then $f_2(B) = D_2$.

Proof. We set $f_2(C) = f_1(C)$ for all nonempty subsets C of D_1 and proceed by induction on i, $\alpha_1 \leqslant i < \alpha_2$. Assume that a mapping f_2 has been defined on the set of all nonempty subsets of $\bigcup_{j<i} C_j$, f_2 is a generating mapping on this set and if $i > \alpha_1$, then assertions (1)–(4) (for f_2 defined on $\bigcup_{j<i} C_j$) of the lemma are true.

Now we define a value of $f_2(C)$ for each nonempty subset C of $\bigcup_{j\leqslant i} C_j$. Consider the subsets $\Phi_1 = \{a_j \colon j < \alpha_1\}$ and $\Phi_{2,i} = \{a_j \colon \alpha_1 \leqslant j < i\}$ ($\Phi_{2,i} = \emptyset$ in the case $i = \alpha_1$) of the set Φ. If $i > \alpha_1$, then on the set $\Phi_{2,i}$ we introduce the ordering of the set $W(\alpha_1)$ (or of an initial segment of it). Hence $\Phi_{2,i} = \{d_j \colon j < \alpha_1$ or $j < \beta$, $\beta \in W(\alpha_1)\}$.

On the set Ψ of all finite nonempty subsets of the set Φ_1 we introduce a linear ordering in the following way. If $\{a_{i_1}, \ldots, a_{i_k}\}$ and $\{a_{j_1}, \ldots, a_{j_s}\}$ are contained in Ψ, where $1 \leqslant i_1 < \cdots < i_k < \alpha_1$ and $1 \leqslant j_1 < \cdots < j_s < \alpha_1$, then we first compare the numbers i_k and j_s. If these numbers are distinct, say $i_k > j_s$, then we write $\{a_{i_1}, \ldots, a_{i_k}\} > \{a_{j_1}, \ldots, a_{j_s}\}$, and if $i_k = j_s$, then we compare i_{k-1} and j_{s-1}, and so on. Finally, if $k > s$ and $i_k = j_s, \ldots, i_{k-s+1} = j_1$, then we define $\{a_{i_1}, \ldots, a_{i_k}\} > \{a_{j_1}, \ldots, a_{j_s}\}$.

If C is a finite subset of $\bigcup_{j\leqslant i} C_j$ and $C \subseteq G_\mu$ for some $\mu \in I$, then we set $f_2(C) = \mathrm{gp}\{C\}\backslash\{1\}$. Let $a_i \in G_\nu$, $\nu \in I$, and let $\{a_{i_1}, \ldots a_{i_k}\}$ be the minimal element of the set Ψ such that $f_2(\{a_i, a_{i_1}, \ldots, a_{i_k}\})$ has not been defined yet. Then we set

$f_{2,1}(\{a_i, a_{i_1}, \ldots, a_{i_k}\}) = \bigcup f_2(E)$, where E runs through all nonempty subsets of the set $A = \{a_i, a_{i_1}, d_{i_1}, \ldots, a_{i_k}, d_{i_k}, a_1, d_1, a_2, d_2\}$ for which the values of the mapping f_2 have been defined by this time, where if an element d_{i_t} of the set $\Phi_{2,i}$ does not exist, then it is not contained in A (for example, the value of $f_2(\{a_{i_1}, \ldots, a_{i_k}, a_1, a_2\})$ has already been defined).

Assume that we have defined $f_{2,m}(\{a_i, a_{i_1}, \ldots, a_{i_k}\})$, $m \geqslant 1$. We denote by Λ_m the set of elements of $\bigcup_{j \leqslant i} C_j$ which are contained in $f_{2,m}(\{a_{i_1}, \ldots, a_{i_k}\})$, and let Υ_m be the set of indexes of all elements of $\Lambda_m \cap (\Phi_1 \cup \Phi_{2,i})$, where the set $\Phi_{2,i}$ is taken with the ordering of $W(\alpha_1)$ (or of an initial segment of it). Then we define

$$\Lambda'_m = \Lambda_m \cup \{d_j\colon\ j \in \Upsilon_m\} \cup \{a_j\colon\ j \in \Upsilon_m\},$$

where if an element d_j of the set $\Phi_{2,i}$ does not exist, then it is not contained in Λ'_m. Now we define $f_{2,m+1}(\{a_i, a_{i_1}, \ldots, a_{i_k}\}) = \bigcup f_2(E)$, where E runs through all finite subsets of the set Λ'_m for which the values of the mapping f_2 have been defined by this time. Finally, we set $f_2(\{a_i, a_{i_1}, \ldots, a_{i_k}\}) = \bigcup_{m \geqslant 1} f_{2,m}(\{a_i, a_{i_1}, \ldots, a_{i_k}\})$. If $\{c_1, \ldots, c_s\} \subseteq f_2(\{a_i, a_{i_1}, \ldots, a_{i_k}\})$ and $f_2(\{c_1, \ldots, c_s\})$ has not been defined yet, then we set $f_2(\{c_1, \ldots, c_s\}) = f_2(\{a_i, a_{i_1}, \ldots, a_{i_k}\})$.

Claim. If $\{a_i, a_{i_1}, \ldots, a_{i_k}\} \nsubseteq G_\nu$ and $a_j \in f_2(\{a_i, a_{i_1}, \ldots, a_{i_k}\})$, $j < \alpha_1$, then $d_j \in f_2(\{a_i, a_{i_1}, \ldots, a_{i_k}\})$ as well, if such a d_j exists, and also if $\{a_i, a_{i_1}, \ldots, a_{i_k}\} \nsubseteq G_\nu$ and $d_j \in f_2(\{a_i, a_{i_1}, \ldots, a_{i_k}\})$ for some $j < \alpha_1$, then $a_j \in f_2(\{a_i, a_{i_1}, \ldots, a_{i_k}\})$.

(It follows from the definitions of $f_2(\{a_i, a_{i_1}, \ldots, a_{i_k}\})$ and the sets Λ'_m, $m \geqslant 1$.)

Now we verify that if

$$\{a_i, a_{i_1}, \ldots, a_{i_k}\} \nsubseteq G_\nu,$$

then the definition of $f_2(\{a_i, a_{i_1}, \ldots, a_{i_k}\})$ satisfies Definition 4(2) of a generating mapping (see Section 2). In fact, for each $m \geqslant 1$, the set $f_{2,m}(\{a_i, a_{i_1}, \ldots, a_{i_k}\})$ is the union of a countable set of countable sets, and hence is countable as well. Then $f_2(\{a_i, a_{i_1}, \ldots, a_{i_k}\})$ is a countable set. Let E be a finite subset of $f_2(\{a_i, a_{i_1}, \ldots, a_{i_k}\})$. Then there exists $m \geqslant 1$ such that $E \subseteq \Lambda_m$, and either $f_2(E) \subseteq f_{2,m+1}(\{a_i, a_{i_1}, \ldots, a_{i_k}\})$ or $f_2(E) = f_2(\{a_i, a_{i_1}, \ldots, a_{i_k}\})$. And finally, $a_i \in f_2(\{a_i\}) \subseteq f_{2,1}(\{a_i, a_{i_1}, \ldots, a_{i_k}\})$ and $a_{i_t} \in f_2(\{a_{i_t}\}) \subseteq f_{2,1}(\{a_i, a_{i_1}, \ldots, a_{i_k}\})$ for each t, $1 \leqslant t \leqslant k$.

Thus we can define the value of $f_2(\{a_i, a_{i_1}, \ldots, a_{i_k}\})$ for each $\{a_{i_1}, \ldots, a_{i_k}\} \in \Psi$. Now we show that, in fact, we have defined f_2 on all finite subsets E of $\bigcup_{j \leqslant i} C_j$. If $E \subseteq \bigcup_{j<i} C_j$, then $f_2(E)$ has been defined by the induction hypothesis. If this is not the case, it follows from Definition 8 of a regular decomposition of Ω that there is a subset $\{a_{j_1}, \ldots, a_{j_s}\}$, $s \geqslant 0$, of $\Phi_1 \cup \Phi_{2,i}$ such that $\{a_i, a_{j_1}, \ldots, a_{j_s}\}' \supseteq E$. We may assume that $\{a_{j_1}, \ldots, a_{j_r}\} \subseteq \Phi_{2,i}$, $0 \leqslant r \leqslant s$, and $\{a_{j_{r+1}}, \ldots, a_{j_s}\} \subseteq \Phi_1$. Hence if we consider the ordering of the set $W(\alpha_1)$ (or of an initial segment of it) on the set $\Phi_{2,i}$, then we have that $a_{j_t} = d_{l_t}$, $l_t < \alpha_1, t \leqslant r$, and by Claim and Definition 4(2),

$$f_2\big(\{a_i, a_{l_1}, \ldots, a_{l_r}, a_{j_{r+1}}, \ldots, a_{j_s}, a_1, a_2\}\big) \supseteq \{a_i, a_{j_1}, \ldots, a_{j_s}\}' \supseteq E.$$

Therefore, the value of $f_2(E)$ has already been defined.

Completing the definition of f_2 on all nonempty subsets of the set $\bigcup_{j\leqslant i} C_j$ (using conditions (1) and (3) in Definition 3 of a generating mapping), we obtain that f_2 is a generating mapping on $\bigcup_{j\leqslant i} C_j$. Assertions (1)–(3) of the lemma follow immediately from the definition of f_2.

Let $b_1 \in C_i$ and $b_2 \in C_j$, where $j < i$, and let $\{a_{j_1}, \ldots, a_{j_s}\}$ be the minimal element of Ψ such that $\{b_1, b_2, a_1, a_2\} \subseteq f_2(\{a_i, a_{j_1}, \ldots, a_{j_s}\})$. By the statement of the lemma, $\{b_1, b_2, a_1, a_2\} \nsubseteq G_\nu$, then also $\{a_i, a_{j_1}, \ldots, a_{j_s}\} \nsubseteq G_\nu$, and it follows from the definition of the mapping f_2 that $f_2(\{b_1, b_2, a_1, a_2\}) = f_2(\{a_i, a_{j_1}, \ldots, a_{j_s}\})$.

Proceeding by induction on i, $\alpha_1 \leqslant i < \alpha_2$, we obtain a generating mapping f_2 on the set D_2. It remains to prove assertion (5) of the lemma.

Let B be a subset of D_2 such that $B \nsubseteq G_\mu$ for each $\mu \in I$ and $|B| = m^+$. Then by assertion (3) of the lemma, $a_1, a_2 \in f(B)$, and it follows from assertion (4) of the lemma that we may assume $B \subseteq \Phi_2 = \{a_j\colon 1 \leqslant j < \alpha_2\}$. Let b be an arbitrary element of D_2. Then $b \in C_l$ for some $l < \alpha_2$. As was noted in the proof of Lemma 9, $C_l \subseteq \mathrm{gp}\{(\Phi_1 \cup \Phi_{2,l+1}) \cap G_\nu\}$ for some $\nu \in I$, hence $|C_l| \leqslant m$ and there exists $i \in W(\alpha_2)\backslash W(\alpha_1)$ such that $i > l$, $a_i \in B$ and a set $T_i = \{a_s\colon a_s \in B,\ s < i\}$ is of cardinality m. Applying assertions (4) and (3) of the lemma to all subsets $\{a_i, a_s, a_1, a_2\}$, where $a_s \in T_i$, we obtain that

$$f_2(\{a_i, a_1, a_2\} \cup T_i) = f_2(\{a_i\} \cup C),$$

where $C \subseteq D_1$, $|C| = m$ and $a_1, a_2 \in C$. By the statement of the lemma, we may assume that $f_2(C) = D_1$, hence $f_2(\{a_i\} \cup C) = f_2(\{a_i\} \cup D_1)$. It follows from Definition 8 of a regular decomposition of Ω that there is a finite subset E of $\Phi_1 \cup \Phi_{2,i}$ such that $b \in E'$. Then by the definition of the mapping f_2 and Claim,

$$b \in E' \subseteq f_2(\{a_i\} \cup E) \subseteq f_2(\{a_i\} \cup D_1) \subseteq f_2(B).$$

Thus $f_2(B) = D_2$, which completes the proof of the lemma. □

Proof of Theorem D. Let $\{C_i\colon C_i \in 2^\Omega\backslash\{\emptyset\},\ i \in W(\omega_n)\}$ be a regular decomposition of the set $\Omega = \Omega^1\backslash\{1\}$ defined by a set $\Phi = \{a_i\colon i \in W(\omega_n)\}$ such that $\{a_1, a_2\} \nsubseteq G_\mu$ for each $\mu \in I$. (It is obvious that such a regular decomposition exists.) We set $D_t = \bigcup_{j<\omega_t} C_j$, $1 \leqslant t \leqslant n$. A generating mapping f on Ω is defined in the following way: we define f on $2^{D_1}\backslash\{\emptyset\}$ in the same way as it was done in Lemma 9, and if $n > 1$, then we use Lemma 10 to complete the definition of f successively on $2^{D_2}\backslash\{\emptyset\}, \ldots, 2^{D_n}\backslash\{\emptyset\}$.

Let H be the trivial group and all homomorphisms $g_\mu\colon G_\mu \to H$, $\mu \in I$, are also trivial. Then Theorem B applies to Ω^1 and f and yields a simple group $G = \mathrm{gp}\{\Omega^1\}$. Let M be a subgroup of G such that $|M| = \aleph_n$ and let M be not contained in a subgroup conjugate in G to some G_μ, $\mu \in I$. By Theorem B(3), M is conjugate in G to a subgroup $\mathrm{gp}\{C\}$ for some $C \subseteq \Omega$ such that $|C| = \aleph_n$ and $C \nsubseteq G_\mu$ for each $\mu \in I$. By Lemmas 9(4) and 10(5), $f(C) = \Omega$, hence $M = G$, by Theorem B(4), as required. □

Proof of Theorem E. If the group M is countable, then it is sufficient to take $\{G_\mu\}_{\mu\in W(\omega_1)}$ to be a set of groups isomorphic to M and G as the group in Theorem D.

If the group M is uncountable, then let β_1, β_2 be the minimal ordinal numbers corresponding to the cardinal numbers α and α^+, respectively, let $\{G_\mu\}_{\mu \in W(\beta_2)}$ be a set of groups isomorphic to M, and let Ω^1 be the free amalgam of the groups G_μ, $\mu < \beta_2$. It is easy to see that the set $\Omega = \Omega^1 \backslash \{1\}$ has a regular decomposition $\{C_i\colon C_i \in 2^\Omega \backslash \{\emptyset\}, i \in W(\beta_2)\}$ defined by a set Φ such that $D_1 = \bigcup_{i<\beta_1} C_i = G_1 \backslash \{1\}$, since we may assume that $\Omega = \{b_j\colon j \in W(\beta_2)\}$, where $\{b_j\colon j \in W(\beta_1)\} = G_1 \backslash \{1\}$, and repeat the construction of a regular decomposition of Ω given after Definition 8. We set $f(B) = \mathrm{gp}\{B\} \backslash \{1\}$ for each $B \in 2^{D_1} \backslash \{\emptyset\}$ and complete the definition of a generating mapping f on Ω using Lemma 10.

Theorem B (with the group H and all homomorphisms g_μ, $\mu \in W(\beta_2)$, trivial) applies to Ω and f and yields a simple group $G = \mathrm{gp}\{\Omega^1\}$ without involutions. Let K be a subgroup of G of cardinality α^+. Every group G_μ, $\mu < \beta_2$, is of cardinality α, then by Theorem B(3), K is conjugate in G to a subgroup $\mathrm{gp}\{C\}$, where $C \subseteq \Omega$, $|C| = \alpha^+$ and $C \nsubseteq G_\mu$ for each $\mu < \beta_2$. By the hypothesis of Theorem E, G_1 is a Jonsson group, hence $f(B) = D_1$ for each subset B of D_1 with $|B| = \alpha$. Then by Lemma 10(5), $f(C) = \Omega$ and $K = G$, by Theorem B(4), which completes the proof of Theorem E. □

Acknowledgements

This work was initiated when the second author visited the University of Wollongong in March 1994. He thanks the Faculty of Informatics and the Mathematical Analysis Research Group for warm hospitality.

References

[1] C.C. Chang and H.J. Keisler, Model Theory (North-Holland, Amsterdam, 1973).
[2] T. Isiwata, On the connectedness of locally compact groups, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A 5 (1955) 8–13.
[3] Kourovka, Notebook: Unsolved Problems of Group Theory (Inst. Math. Siberian Dep. Russian Acad. Sci., Novosibirsk, 12th ed., 1992).
[4] T. Matumoto, Any group is represented by outer automorphism group, Hiroshima Math. J. 19 (1989) 209–219.
[5] S.A. Morris, A characterization of the topological group of real numbers, Bull. Austral. Math. Soc. 34 (1986) 473–475.
[6] S.A. Morris, The circle group, Bull. Austral. Math. Soc. 36 (1987) 279–282.
[7] S.A. Morris and V.N. Obraztsov, Embedding free amalgams of discrete groups in nondiscrete topological groups, to appear.
[8] V.N. Obraztsov, Some infinite groups, Sibirsk. Mat. Zh. 32 (1) (1991) 99–106 (in Russian); English translation: Siberian Math. J. 32 (1) (1991) 79–85.
[9] V.N. Obraztsov, Embedding schemes for groups and some applications, deposited VINITI 8 February 1990, No. 724–790.
[10] V.N. Obraztsov, A new embedding scheme for groups and some applications, J. Austral. Math. Soc. Ser. A 61 (1996) 267–288.
[11] A.Yu. Ol'shanskii, A remark on a countable nontopologized group, Vestnik Moskov. Univ. Ser. I Mat. Mekh. (3) (1980) 103.

[12] A.Yu. Ol'shanskii, Geometry of Defining Relations in Groups (Nauka, Moscow, 1989) (in Russian); English translation: Dordrecht, 1991.
[13] L.C. Robertson and B.M. Schreiber, The additive structure of integer groups and p-adic number fields, Proc. Amer. Math. Soc. 19 (1968) 1453–1456.
[14] S. Shelah, On a problem of Kurosh, Jonsson groups, and applications, in: Word Problems, Vol. II (North-Holland, Amsterdam, 1980) 373–394.