THE FREE ABELIAN TOPOLOGICAL GROUP AND THE FREE LOCALLY CONVEX SPACE ON THE UNIT INTERVAL

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Dedicated to the memory of Eli Katz

ABSTRACT

We give a complete description of the topological spaces X such that the free abelian topological group $A(X)$ embeds into the free abelian topological group $A(I)$ on the closed unit interval. In particular, the free abelian topological group $A(X)$ on any finite-dimensional compact metrizable space X embeds into $A(I)$. To obtain our description, we study similar embeddings of the free locally convex spaces and continuous surjections between the spaces of continuous functions with the pointwise topology. Proofs are based on the classical Kolmogorov's Superposition Theorem.

1. *Introduction*

The following natural question arises as a part of the search for a topologized version of the Nielsen–Schreier subgroup theorem. Let X and Y be completely regular topological spaces; in which cases can the free (free abelian) topological group on X be embedded as a topological subgroup into the free (free abelian) topological group on Y? This problem has been studied for a long time [**4**, **10**, **11**–**16**, **21**–**23**, **25**, **27**, **36**], ever since it became clear that in general a topological subgroup of a free (free abelian) topological group need not be topologically free [**8**, **4**, **10**]. Recently a complete answer was obtained in the case where X is a subspace of Y and the embedding of free topological groups extends the embedding of spaces [**31**]. However, we are interested in the existence of an embedding which is not necessarily a 'canonical' one. Among the most notable achievements, there are certain sufficient conditions for a subgroup of a free topological group to be topologically free [**4**, **22**] and the following results.

THEOREM 1.1 [**13**]. *If X is a closed topological subspace of the free topological group $F(I)$, then the free topological group $F(X)$ is a closed topological subgroup of $F(I)$, where I is the closed unit interval.*

COROLLARY 1.2 [**22**]. *If X is a finite-dimensional metrizable compact space, then $F(X)$ is a closed topological subgroup of $F(I)$.*

The abelian case proved to be more difficult, and the following is the strongest result known to date.

THEOREM 1.3 [**11**]. *If X is a countable* CW-*complex of dimension n, then the free abelian topological group on X is a closed subgroup of the free abelian topological group on the closed ball B^n.*

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COROLLARY 1.4 [**12**]. *The free abelian topological group $A(\mathbb{R})$ embeds into $A(I)$ as a closed topological subgroup.*

It is known [**28**] that the covering dimension of any two free topological bases in a free (abelian) topological group is the same; this result is similar to the well-known property of free bases of a discrete free (abelian) group having the same cardinality, called the rank of the group. Since the rank of a subgroup of a free abelian group cannot exceed the rank of the group itself, it was conjectured [**14**, **20**] that the dimension of a topological basis of a topologically free subgroup of a free abelian topological group $A(X)$ cannot exceed $\dim X$. It even remained unclear whether the group $A(I^2)$ embeds into $A(I)$ [**14**].

In this paper we prove that if X is a completely regular space then the free abelian topological group $A(X)$ embeds into $A(I)$ as a topological subgroup if and only if X is a submetrizable k_ω-space such that every compact subspace of X is finite-dimensional. Another characterization: X is homeomorphic to a closed topological subspace of the group $A(I)$ itself. In particular, if X is a compact metrizable space of finite dimension, then $A(X)$ embeds into $A(I)$. Thus, the analogy with the non-abelian case is complete.

We also study the problem of embedding the free locally convex space $L(X)$ into the free locally convex space $L(I)$. In particular, we show that such an embedding exists for every compact metrizable finite-dimensional space X.

Our results provide answers to a number of open problems from [**20**, **14**, **38**].

2. *Preliminaries*

The following concept goes back to [**19**, **8**]; a detailed up-to-date survey is [**20**].

DEFINITION 2.1. Let X be a topological space. The (*Markov*) *free abelian topological group* on X is a pair consisting of an abelian topological group $A(X)$ and a continuous mapping $i: X \to A(X)$ such that every continuous mapping f from X to an abelian topological group G gives rise to a unique continuous homomorphism $\bar{f}: A(X) \to G$ with $f = \bar{f} \circ i$.

The free abelian topological group $A(X)$ always exists and is essentially unique. If X is a completely regular T_1 topological space, then $A(X)$ is Hausdorff and algebraically free over the set X and the mapping i is a topological embedding, $i: X \hookrightarrow A(X)$ [**19**, **8**, **20**]. Since all spaces in this paper are completely regular T_1, we shall always identify X with a subspace of $A(X)$ in the above canonical way and thus suppress the mapping i altogether in our notation.

All locally convex spaces (LCS, for short) in this paper are real.

DEFINITION 2.2 [**19**, **1**, **30**, **6**, **7**, **35**]. Let X be a topological space. The *free locally convex space* on X is a pair consisting of a locally convex space $L(X)$ and a continuous mapping $i: X \to L(X)$ such that every continuous mapping f from X to a locally convex space E gives rise to a unique continuous linear operator $\bar{f}: L(X) \to E$ with $f = \bar{f} \circ i$.

The free locally convex space $L(X)$ always exists and is essentially unique. If X is a completely regular topological space, then $L(X)$ is separated, the set X forms a Hamel basis for $L(X)$, and the mapping i is a topological embedding, $i: X \hookrightarrow L(X)$ [**30**,

6, **7**, **35**]. In the present paper we identify X with a topological subspace of $L(X)$. One well-known example is the vector space $\mathbb{R}^\infty$ equipped with the strongest locally convex topology; algebraically, it is a vector space of countably infinite dimension, and a subset V of $\mathbb{R}^\infty$ is open if and only if so are all intersections of V with finite-dimensional vector subspaces. The space $\mathbb{R}^\infty$ can be identified with the free locally convex space on a countably infinite space with discrete topology.

The identity mapping $\mathrm{id}_X\colon X \to X$ extends to a canonical continuous homomorphism $i\colon A(X) \to L(X)$. The following result was first announced in [**33**] and later furnished with a complete proof in [**36**].

THEOREM 2.3. *The canonical homomorphism* $i\colon A(X) \hookrightarrow L(X)$ *is an embedding of* $A(X)$ *into the additive topological group of the* LCS $L(X)$ *as a closed additive topological subgroup.*

In what follows, we shall often identify $A(X)$ with a subgroup of $L(X)$ in the above canonical way. Denote by $L_p(X)$ the free locally convex space $L(X)$ endowed with the weak topology.

THEOREM 2.4 [**6**, **7**]. *Let X be a completely regular space. The canonical mapping* $X \hookrightarrow L_p(X)$ *is a topological embedding, and every continuous mapping f from X to a locally convex space E with the weak topology extends uniquely to a continuous linear operator* $\bar{f}\colon L_p(X) \to E$.

The weak dual space to $L(X)$ is canonically isomorphic to the space $C_p(X)$ of all continuous real-valued functions on X with the topology of pointwise (simple) convergence. The spaces $L_p(X)$ and $C_p(X)$ are in duality. Denote by $C_k(X)$ the space of continuous functions endowed with the compact-open topology. A topological space X is called *Dieudonné complete* [**5**] if its topology is induced by a complete uniformity. For example, every Lindelöf space is Dieudonné complete.

THEOREM 2.5 (Arhangel'skiĭ [**3**]). *Let X and Y be Dieudonné complete spaces. If a linear mapping* $C_p(X) \to C_p(Y)$ *is continuous, then it is continuous as a mapping* $C_k(X) \to C_k(Y)$.

The space $L(X)$ admits a canonical continuous monomorphism

$$L(X) \hookrightarrow C_k(C_k(X)).$$

THEOREM 2.6 [**6**, **7**, **35**]. *If X is a k-space, then the monomorphism* $L(X) \hookrightarrow C_k(C_k(X))$ *is an embedding of locally convex spaces.*

We shall also use the following fact which can be easily extracted from [**34**].

THEOREM 2.7. *Let* $T\colon C_p(Y) \to C_p(X)$ *be a linear continuous surjection. If Y is metrizable compact, then X is also metrizable compact.*

Let X be a topological space. A collection of continuous functions $h_1, \ldots, h_m$ on X, assuming their values in the closed unit interval $I = [0, 1]$ is called *basic* [**26**, **32**] if every real-valued continuous function f on X can be represented as a sum $\sum_{i=1}^m g_i \circ h_i$ of compositions of basic functions with some continuous functions $g_i \in C(I)$.

2.8. Kolmogorov's Superposition Theorem [**17**]. *The finite-dimensional cube I^n has a finite basic family of continuous real-valued functions.*

Let us recall that for compact metrizable spaces all three main concepts of dimension (the covering, the small inductive and the large inductive ones) coincide [**5**]. The following result is of crucial importance for us; it is a corollary of Kolmogorov's Superposition Theorem, the Menger–Nöbeling Theorem on embeddability of separable metric spaces of dimension at most n into $\mathbb{R}^{2n+1}$, and the Tietze–Urysohn Extension Theorem [**5**].

Corollary 2.9 (Ostrand [**26**]). *Let X be a finite-dimensional compact metrizable space. Then there exists a finite basic family of continuous functions on X.*

For an exact upper bound on the cardinality of a basic family of continuous functions on a space X of dimension n, see [**32**]; however, we do not need it.

All topological spaces in this paper are completely regular T_1. A topological space X is called a *k_ω-space* [**18**, **12–16**] if there exists what is called a *k_ω-decomposition* $X = \bigcup_{n \in \mathbb{N}} X_n$, where all the X_n are compact, $X_n \subset X_{n+1}$ for $n \in \mathbb{N}$, and a subset $A \subset X$ is closed if and only if all intersections $A \cap X_n$ for $n \in \mathbb{N}$, are closed. Being Lindelöf, every k_ω-space is therefore Dieudonné complete. A topological space X is called *submetrizable* if it admits a continuous one-to-one mapping into a metrizable space.

3. *Auxiliary constructions*

Lemma 3.1. *Consider a commutative diagram of Banach spaces and surjective linear mappings:*

$$\begin{array}{ccccccccccc}
E_1 & \xleftarrow{r_1} & E_2 & \xleftarrow{r_2} & E_3 & \xleftarrow{r_3} & \cdots & \xleftarrow{r_{n-1}} & E_n & \xleftarrow{r_n} & \cdots \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_3} & & & & \downarrow{\scriptstyle \pi_n} & & \\
F_1 & \xleftarrow{q_1} & F_2 & \xleftarrow{q_2} & F_3 & \xleftarrow{q_3} & \cdots & \xleftarrow{q_{n-1}} & F_n & \xleftarrow{q_n} & \cdots
\end{array}$$

Denote by $E = \varprojlim E_n$ and $F = \varprojlim F_n$, the Fréchet spaces projective limits of the corresponding inverse sequences, and by $\pi \colon E \to F$ the projective limit of the mappings π_n for $n \in \mathbb{N}$. Then every compact subspace $K \subset F$ is the image under the mapping π of a compact subspace of E.

Proof. Let K be a compact subspace of F. Set $K_n = q_n(K)$ for all $n \in \mathbb{N}$. According to the Michael Selection Theorem [**37**, Theorem 1.4.9], there exists a compact subspace $C_1 \subset E_1$ such that $\pi_1(C_1) = K_1$. Suppose now that for all $k \leqslant n$ we have chosen compact subspaces $C_k \subset E_k$ such that $\pi_k(C_k) = K_k$ and $r_{k-1}(C_k) = C_{k-1}$. Consider the mapping $\langle r_n, \pi_{n+1} \rangle \colon x \mapsto (r_n(x),\ \pi_{n+1}(x))$ from E_{n+1} to $E_n \times F_{n+1}$. The subset $Q_n = \{(y, z) \colon y \in C_n,\ z \in K_{n+1}, q_n(z) = \pi_n(y)\}$ of the space $E_n \times F_{n+1}$ is compact, and is contained in the Banach space image of the continuous linear mapping $\langle r_n, \pi_{n+1} \rangle$. Therefore, by the Michael Selection Theorem, there exists a compact subset $C_{n+1} \subset E_{n+1}$ such that $\langle r_n, \pi_{n+1} \rangle (C_{n+1}) = Q_n$. Consequently, $r_n(C_{n+1}) = C_n$, and $q_{n+1}(C_{n+1}) = K_{n+1}$, which completes the recursion step.

Finally, put $C = \varprojlim C_n$; this subset of E is compact, and the property $K \subset \varprojlim K_n$ implies that $\pi(C) = K$.

Lemma 3.2. *Let X and Y be k_ω-spaces. Let $h: L_p(X) \to L_p(Y)$ be an embedding of locally convex spaces. Then h is also an embedding of the locally convex space $L(X)$ into $L(Y)$.*

Proof. As a corollary of the Hahn–Banach theorem, the dual linear map $h^*: C_p(Y) \to C_p(X)$ to the embedding h is a continuous homomorphism onto. Theorem 2.5 says that h^* remains continuous with respect to the compact-open topologies on both spaces, and by virtue of the Open Mapping Theorem, $h^*: C_k(Y) \to C_k(X)$ is open. Since for every compact subset $C \subset X$ the elements of the image $h(C)$ are contained in the linear span of a compact subset of Y [**3**], one can choose k_ω decomposition $X = \bigcup_{n=1}^\infty X_n$ and $Y = \bigcup_{n=1}^\infty Y_n$ in such a way that, for every $n \in \mathbb{N}$, one has $h(\mathrm{sp}\, X_n) \subset \mathrm{sp}\, Y_n$. It is easy to see that the restrictions mappings $r_n: C_k(Y_{n+1}) \to C_k(Y_n)$ and $q_n: C_k(X_{n+1}) \to C_k(X_n)$ are continuous onto, and that $C(Y) = \varprojlim C_k(Y_n)$ and $C(X) = \varprojlim C_k(X_n)$. Denote by π_n, for each $n \in \mathbb{N}$, the restriction $h^*|_{C_k(Y_n)}$. The conditions of Lemma 3.1 are fulfilled, and therefore every compact subset $K \subset C_k(X)$ is an image under the mapping h^* of a suitable compact subset of $C_k(Y)$. Hence, the continuous linear map h^{**} dual to h^* from the space $C_k(C_k(X))$ to $C_k(C_k(Y))$ is an embedding of $C_k(C_k(X))$ into $C_k(C_k(Y))$ as a locally convex subspace. Since h coincides with the restriction of h^{**} to $L(X)$, Theorem 2.6 implies that it is an embedding of $L(X)$ into $L(Y)$.

Lemma 3.3. *Let X be a compact space and let Y be a closed subspace of X. Denote by π the quotient mapping from X to X/Y. Let the f_k for $k = 1, \ldots, n$ be continuous functions on X such that their restrictions to Y form a basic family for Y, and let the g_i for $i = 1, \ldots, m$ be a basic family of functions on X/Y. Then the family of functions $f_1, \ldots, f_n, g_1 \circ \pi, \ldots, g_m \circ \pi$ is basic for X.*

Proof. Let $f: X \to \mathbb{R}$ be a continuous function. For a family of continuous functions $h_1, \ldots, h_n \in C(I)$, the restriction $f|_Y$ is represented as $\sum_{k=1}^n h_k \circ (f_k|_Y) = (\sum_{k=1}^n h_k \circ f_k)|_Y$. Denote by $g: X \to \mathbb{R}$ the continuous function $f - \sum_{k=1}^n h_k \circ f_k$; since the restriction $g|_Y \equiv 0$, the function g factors through the mapping π, that is, there exists a continuous function $h: X/Y \to I$ with $g = h \circ \pi$. For some collection $s_1, \ldots, s_m$ of continuous functions on I one has $h = \sum_{i=1}^m s_i \circ g_i$, which means that $g = \sum_{i=1}^m s_i \circ g_i \circ \pi$. Finally, one has

$$f = \sum_{k=1}^n h_k \circ f_k + \sum_{i=1}^m s_i \circ g_i \circ \pi,$$

as desired.

Lemma 3.4. *Let X be a submetrizable k_ω-space with k_ω-decomposition $X = \bigcup_{n \in \mathbb{N}} X_n$ such that every subspace X_n is finite-dimensional. Then there exists an embedding of locally convex spaces $\bar{F}: L_p(X) \hookrightarrow L_p(Y)$, where Y is the disjoint sum of countably many copies of the closed unit interval I, such that $\bar{F}(A(X)) \subset A(Y)$.*

Proof. Let $X = \bigcup_{n \in \mathbb{N}} X_n$ be a k_ω decomposition of X with $X_n \subset X_{n+1}$ for all $n \in \mathbb{N}$. Since every X_n for $n \in \mathbb{N}$ is a finite-dimensional metrizable compact space, then for any $n\in\mathbb{N}$ so is the quotient space X_{n+1}/X_n, and one can choose inductively, using Ostrand's Corollary 2.9 and Lemma 3.3, a countable family of continuous functions $f_{n,i}$ for $n\in\mathbb{N}$, $i=1,\dots,k_n$ with $k_n\in\mathbb{N}$ from X to I such that for each $n\in\mathbb{N}$ the following are true:

(1) the collection $f_{m,i}$, with $i=1,\dots,k_m$, $m=1,\dots,n$, is basic for X_n;

(2) $f_{n+1}|_{X_n}\equiv 0$ for all $i=1,\dots,k_{n+1}$.

Denote the above family of functions $f_{n,i}$ by $\mathscr{F}$, and let $Y=\oplus_{f\in\mathscr{F}} I_f$ be the disjoint sum of countably many copies of the closed unit interval I. For every $f\in\mathscr{F}$ denote by O_f the left endpoint of the closed interval I_f, regarded as an element of the free abelian group $A(Y)$.

Define a mapping, F, from X to the free abelian group $A(Y)$ by letting

$$F(x)=\sum_{i=1,\dots,k_1} f_{1,i}(x)+\sum_{n\geqslant 2,\, i=1,\dots,k_n}(f_{n,i}(x)-O_{n,i})$$

for each $x\in X$. The mapping F is well-defined, because in the second sum all but finitely many terms are vanishing in the free abelian group $A(Y)$ for every $x\in X$. The restriction of F to every X_n is continuous, which follows from the continuity of each mapping $f_{m,i}\colon X_n\to I_{f_{m,i}}\subset Y$ with $m\leqslant n$, $i=1,\dots,k_m$ and the continuity of subtraction and addition in $A(Y)$. Therefore the mapping $F\colon X\to A(Y)$ is continuous. If viewed as a continuous mapping from X to the locally convex space $L_p(Y)$, it extends to a continuous linear operator $\bar{F}\colon L_p(X)\to L_p(Y)$.

Let $h\colon X\to\mathbb{R}$ be a continuous function. We shall show that there exists a continuous linear functional $\bar{h}$ on the linear subspace $\bar{F}(L_p(X))$ such that $\bar{h}\circ F|_X=h$. This would mean that $\bar{F}(L_p(X))$ is isomorphic to $L_p(X)$, as desired.

Construct recursively, making use of the Properties 1 and 2 above, a countable family of continuous functions $h_{n,i}$, with $i=1\dots,k_n, n\in\mathbb{N}$, from I to $\mathbb{R}$ such that for every $n\in\mathbb{N}$ and for all $x\in X_n$, one has

$$h(x)=\sum_{i=1\dots,k_m,\, m\leqslant n}(h_{m,i}\circ f_{m,i})(x).$$

Let us recall that $f_{n,i}|_{X_1}\equiv 0$ for all $n\geqslant 2$ and $i=1,\dots,k_n$. It is easy to deduce inductively from this fact that for any $n\geqslant 2$ one has

$$\sum_{i=1\dots,k_n} h_{n,i}(0)=0.$$

Define a continuous mapping H from Y to I by letting $H(y)=h_{n,i}(y)$ whenever $y\in I_{n,i}$, $n\in\mathbb{N}$. Extend H to a continuous linear functional $\bar{H}\colon L_p(Y)\to\mathbb{R}$ and denote its restriction to $\bar{F}(L_p(X))$ *by* $\bar{h}$. We claim that $\bar{h}\circ F|_X=h$, or, which is the same, that for every $n\in\mathbb{N}$ one has $\bar{h}\circ F|_{X_n}=h$. Indeed, for an arbitrary $x\in X_n$ one has:

$$(\bar{h}\circ F)(x)=\bar{H}(F(x))=\bar{H}\Bigg(\sum_{i=1,\dots,k_1} f_{1,i}(x)+\sum_{\substack{2\leqslant m\leqslant n,\\ i=1,\dots,k_n}}(f_{m,i}(x)-O_{m,i})\Bigg)$$

$$=\sum_{i=1,\dots,k_1} H(f_{1,i}(x))+\sum_{\substack{2\leqslant m\leqslant n,\\ i=1,\dots,k_n}} H(f_{m,i}(x)-O_{m,i})$$

$$=\sum_{\substack{i=1,\dots,k_m,\\ m\leqslant n}}(h_{m,i}\circ f_{m,i})(x)-\sum_{\substack{2\leqslant m\leqslant n,\\ i=1,\dots,k_n}} h_{m,i}(0)=h(x)-0=h(x).$$

4. *Main results*

THEOREM 4.1. *For a completely regular T_1-space X the following are equivalent:*

(i) *the free abelian topological group $A(X)$ embeds into $A(I)$ as a topological subgroup;*

(ii) *the free topological group $F(X)$ embeds into $F(I)$ as a topological subgroup;*

(iii) *X is homeomorphic to a closed topological subspace of $A(I)$;*

(iv) *X is homeomorphic to a closed topological subspace of $F(I)$;*

(v) *X is homeomorphic to a closed topological subspace of $\mathbb{R}^\infty$;*

(vi) *X is a k_ω-space such that every compact subspace of X is metrizable and finite-dimensional;*

(vii) *X is a submetrizable k_ω-space such that every compact subspace of X is finite-dimensional.*

Proof. (i) $\Rightarrow$ (iii) Since the space X is Lindelöf (as a subspace of $A(I)$, see [**2**]) and hence Dieudonné complete, the group $A(X)$ is complete in its two-sided uniformity [**33**] and therefore closed in $A(I)$; but X is closed in $A(X)$.

(ii) $\Leftrightarrow$ (iv) See Theorem 1.1.

(iii) $\Leftrightarrow$ (v) $\Leftrightarrow$ (iv) The result of Zarichnyĭ [**39**] states that the free topological group $F(I)$ and the free abelian topological group $A(I)$ are homeomorphic to open subsets of $\mathbb{R}^\infty$. Now it follows that X is homeomorphic to a closed subset of an open subset of $\mathbb{R}^\infty$. Since X is also k_ω, it is easy to construct a homeomorphism of X with a closed subset of $\mathbb{R}^\infty$ with the help of standard arguments from infinite-dimensional topology.

(v) $\Rightarrow$ (vi) The space $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$ is a k_ω-space such that every compact subspace of it is metrizable and finite-dimensional, and this property is inherited by closed subsets.

(vi) $\Leftrightarrow$ (vii) See [**15**].

(vii) $\Rightarrow$ (i) Let X be a submetrizable k_ω-space such that every compact subspace of X is finite-dimensional. According to Lemma 3.4, there exists an embedding of locally convex spaces $\bar{F}\colon L_p(X) \hookrightarrow L_p(Y)$, where Y is the disjoint sum of countably many copies of the closed unit interval I, such that $\bar{F}(A(X)) \subset A(Y)$. By virtue of Lemma 3.2, $\bar{F}$ is also an embedding of locally convex spaces $L(X) \hookrightarrow L(Y)$. Its restriction to $A(X)$ is an embedding of topological groups (Theorem 2.3). Now apply Theorem 1.3.

THEOREM 4.2. *For a completely regular T_1-space X the following conditions are equivalent.*

(i) *The free locally convex space $L(X)$ embeds into $L(I)$ as a locally convex subspace.*

(ii) *The free locally convex space with the weak topology $L_p(X)$ embeds into $L_p(I)$ as a locally convex subspace.*

(iii) *The linear topological space $C_p(X)$ is an image of $C_p(I)$ under a linear continuous surjection.*

Proof. (ii) $\Leftrightarrow$ (iii) These are just the dual forms of the same statement about two locally convex spaces having weak topology.

(iii) $\Rightarrow$ (i) Theorem 2.7 implies that X is metrizable compact, and so it now suffices to apply the previous implication together with Lemma 3.2.

(i) $\Rightarrow$ (iii) This follows from the fact that $C_p(X)$ is the weak dual space to $L(X)$.

A compact space X which can be represented as a countable union of finite-dimensional compact subspaces is called a *countable-dimensional space*.

THEOREM 4.3. *Let X be a completely regular T_1-space such that the free locally convex space $L(X)$ embeds into $L(I)$ as a locally convex subspace. Then X is a metrizable countable-dimensional compactum.*

Proof. The space $L(I)$ is a countable union of closed subspaces $\mathrm{sp}_n(I)$ formed by all words of reduced length at most n over I for $n \in \mathbb{N}$. Since $\mathrm{sp}_n(I)$ is a union of countably many closed subspaces, each of which is homeomorphic to a subspace of the nth Tychonoff power of the space $\mathbb{R} \times [I \oplus (-I) \oplus \{0\}]$ [**2**], the space $\mathrm{sp}_n(I)$ is topologically finite-dimensional. Theorem 2.7 finishes the proof.

THEOREM 4.4. *Let X be a finite-dimensional metrizable compactum. Then the free locally convex space $L(X)$ embeds into $L(I)$ as a locally convex subspace.* (*Equivalently, $C_p(X)$ is an image of $C_p(I)$ under a linear continuous surjection.*)

Proof. As a consequence of Lemmas 3.3 and 3.4, $L_p(X)$ embeds as a locally convex subspace into the free locally convex space in the weak topology over a disjoint sum of finitely many homeomorphic copies of the closed interval. The latter LCS naturally embeds into $L_p(I)$ (and, in fact, is even isomorphic to it). Now Theorem 4.2 is applied.

REMARK 4.5. Surprising as it may seem, in view of Corollary 4.3 the free locally convex space $L(\mathbb{R})$ does not embed into $L(I)$, in spite of the existence of canonical embeddings $A(\mathbb{R}) \hookrightarrow L(\mathbb{R})$ and $A(I) \hookrightarrow L(I)$ and a (non-canonical) embedding $A(\mathbb{R}) \hookrightarrow A(I)$. It is yet another illustration of the well-known fact that not every continuous homomorphism to the additive group of reals from a closed additive subgroup of an (even normable) LCS extends to a continuous linear functional on the whole space. Such misbehaviour is also to blame—at least partly—for an apparent lack of progress in attempts to make the Pontryagin–van Kampen duality work for free abelian topological groups [**24**, **29**].

REMARK 4.6. It was kindly pointed out to the authors by M. Levin and Y. Sternfeld that by developing our technique further one can construct a linear continuous surjection from $C_p(I)$ onto $C_p(X)$, where X is the one-point compactification of the disjoint union of the Euclidean cubers I^n for $n \in \mathbb{N}$. Therefore, the existence of an embedding $L(X) \hookrightarrow L(I)$ does not necessarily imply that the compactum X is finite-dimensional.

On the other hand, the space $L(Q)$, where $Q = I^{\aleph_0}$ is the Hilbert cube, admits no embedding into $L(I)$ in view of Theorem 4.3: the Baire category arguments show easily that Q is not countable-dimensional.

CONJECTURE 4.7. The following conditions are equivalent for a completely regular T_1-space X:

(i) $L(X)$ embeds into $L(I)$ as a locally convex subspace;

(ii) X is a countable-dimensional metrizable compactum.

REMARK 4.8. Another closely related open problem is that of direct characterization of the covering dimension of a completely regular space X in terms of the linear topological structure of the space $C_p(X)$ (motivated by the principal result in [**9**]). For metrizable compacta a description of dimension in the language of basic functions due to Sternfeld [**32**] might be useful.

REMARK 4.9. Our results also provide answers to two problems from the book *Open Problems in Topology* [**38**].

PROBLEM 511. Is $A(I^2)$ topologically isomorphic with a subgroup of $A(I)$?

Yes (cf. Theorem 4.1).

PROBLEM 1046. Assume that $C_p(X)$ can be mapped by a linear continuous mapping onto $C_p(Y)$. Is it true that $\dim Y \leqslant \dim X$? What if X and Y are compact?

No, in both cases (cf. Theorem 4.4).

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