PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 31, No. 2, February 1972

QUOTIENT GROUPS OF TOPOLOGICAL GROUPS WITH NO SMALL SUBGROUPS

SIDNEY A. MORRIS

ABSTRACT. It is shown here that a quotient group of a topological group with no small subgroups *can* have small subgroups.

It is well known that a quotient group of a Lie group is a Lie group, or equivalently ([**2**], [**6**]) that a quotient group of a locally compact group with no small subgroups is a locally compact group with no small subgroups. Irving Kaplansky [**3**] asks: if G is a topological group with no small subgroups and H is a closed normal subgroup of G, is G/H (necessarily) a group with no small subgroups? We answer the question in the negative here.

THEOREM. *If X is any metric space and $F(X)$ is the free abelian topological group on X* [**4**], *then $F(X)$ has no small subgroups.*

PROOF. By the Arens-Eells embedding theorem ([**5**], [**1**]) X can be embedded isometrically in a normed linear space N as a Hamel basis. Let S be the subgroup of (the additive group structure of) N generated by X. We claim that N, and hence S, has no small subgroups. This is seen by noting that the unit ball of N contains no nontrivial subgroups.

Since $F(X)$ is the free abelian topological group on X, there exists a continuous homomorphism f of $F(X)$ onto S which acts identically on X. Using the fact that X is a Hamel basis for N, we see that f is an algebraic isomorphism. Consequently $F(X)$ has no small subgroups.

EXAMPLE. Let X be the cartesian product of a countably-infinite family of topological groups, each of which is topologically isomorphic to the circle group with its usual topology. Clearly X is a topological group with small subgroups.

Since X is a metric space, by the above theorem, $F(X)$ has no small subgroups. Finally we note that, by Theorem 23 of [**4**], X is a quotient group of $F(X)$.

Received by the editors April 19, 1971.

AMS 1970 *subject classifications*. Primary 22A05, 22E15.

Key words and phrases. Topological group with no small subgroups, free abelian topological group, quotient group.

REFERENCES

1. R. F. Arens and J. Eells, Jr., *On embedding uniform and topological spaces*, Pacific J. Math. **6** (1956), 397–403. MR **18,** 406.

2. A. Gleason, *Groups without small subgroups*, Ann. of Math. (2) **56** (1952), 193–212. MR **14,** 135.

3. Irving Kaplansky, *Lie algebras and locally compact groups*, Univ. of Chicago Press, Chicago, 1971.

4. A. A. Markov, *On free topological groups*, Izv. Akad. Nauk SSSR Ser. Mat. **9** (1945), 3–64; English transl., Amer. Math. Soc. Transl. (1) **8** (1962), 195–272. MR **7,** 7.

5. E. Michael, *A short proof of the Arens-Eells embedding theorem*, Proc. Amer. Math. Soc. **15** (1964), 415–416. MR **28** #5421.

6. D. Montgomery and L. Zippin, *Small subgroups of finite-dimensional groups*, Ann. of Math. (2) **56** (1952), 213–241. MR **14,** 135.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32601

Current address: Department of Mathematics, University of New South Wales, Kensington, N.S.W., 2033 Australia