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Open subgroups of free abelian topological groups

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1. *Introduction*

We prove that any open subgroup of the free abelian topological group on a completely regular space is a free abelian topological group. Moreover, the free topological bases of both groups have the same covering dimension. The prehistory of this result is as follows. The celebrated Nielsen–Schreier theorem states that every subgroup of a free group is free, and it is equally well known that every subgroup of a free abelian group is free abelian. The analogous result is not true for free (abelian) topological groups [**1**, **5**]. However, there exist certain sufficient conditions for a subgroup of a free topological group to be topologically free [**2**]; in particular, an open subgroup of a free topological group on a k_ω-space is topologically free. The corresponding question for free abelian topological groups asked 8 years ago by Morris [**11**] proved to be more difficult and remained open even within the realm of k_ω-spaces. In the present paper a comprehensive answer to this question is obtained.

2. *Markov and Graev free abelian topological groups*

Definition 1 [**9**, **10**]. Let X be a topological space. The Markov free abelian topological group on X is a pair consisting of an abelian topological group $A_M(X)$ and a topological embedding $X \hookrightarrow A_M(X)$ such that every continuous mapping f from X to an abelian topological group G extends uniquely to a continuous homomorphism $\bar{f}\colon A_M(X) \to G$. ∎

Definition 2 [**3**]. Let $X = (X, *)$ be a pointed topological space. The Graev free abelian topological group on X is a pair consisting of an abelian topological group $A_G(X)$ and a topological embedding $X \hookrightarrow A_G(X)$ sending $*$ to the zero element such that every continuous mapping f from X to an abelian topological group G with the property $f(*) = 0_G$ extends uniquely to a continuous homomorphism $\bar{f}\colon A_G(X) \to G$. ∎

It was settled by the founders of the theory that both the Markov and the Graev free abelian topological groups exist and are unique for any completely regular (pointed) topological space X, and algebraically they are free abelian groups (for the Markov group the set of free generators is X itself, and for the Graev group it is the set $X\backslash\{*\}$). For different choices of a basepoint $* \in X$, the resulting Graev free abelian topological groups are isomorphic [**3**]. In addition, it is known [**3**] that $A_M(X) \cong A_G(X) \oplus \mathbb{Z}$.

3. *Main results*

PROPOSITION 1. *Let ϵ be an idempotent endomorphism of a free Markov (Graev) topological abelian group, $A(X)$. Then* $\ker \epsilon$ *is a free Markov (respectively, Graev) topological abelian group.*

Proof. Set $K = \ker \epsilon$ and define a continuous homomorphism $\psi: A(X) \to A(X)$ by [redacted]

$\psi^2 = \psi$. We have $\psi(A(X)) = K$ and $\ker \psi = \epsilon(A(Z))$. We set $Z = \psi(X)$ and shall consider the continuous surjective function $\psi | X: X \to Z$. Notice

$$(\forall x \in X)\, \psi(\epsilon(x)) = 0. \tag{1}$$

Now let $\phi: Z \to G$ be a continuous map into a topological abelian group. Then the continuous map $\phi \circ (\psi | X): X \to G$ uniquely extends to a continuous morphism $\mu: FX \to G$; that is,

$$\mu | X = \phi \circ (\psi | X). \tag{2}$$

We define the morphism $\phi': K \to G$ by

$$\phi' = \mu | K. \tag{3}$$

Claim I. $\mu(\epsilon(x)) = 0$ for all $x \in X$.

Proof. By (2) and (1) we have $\mu(\epsilon(x)) = \phi(\psi[\epsilon(x)]) = \phi(1) = 1$ which proves the claim.

Claim II. ϕ' extends ϕ.

Proof. Let $z \in Z$. Then for some $x \in X$ one may write $z = \psi(x) = x - \epsilon(x)$. Now $\phi'(z) = \mu(z)$ by (3) and $\mu(z) = \mu(x - \epsilon(x)) = \mu(x) - \mu(\epsilon(x)) = \mu(x)$ by Claim I. But $\mu(x) = \phi(\psi(x)) = \phi(z)$ by (2). We have seen that $\phi'(z) = \phi(z)$ or all $z \in Z$.

Claim III. Z generates K (as a topological group).

$\ker\beta \cap \operatorname{im}\alpha$ is free as a subgroup of a free abelian group. Since free abelian discrete groups are projective in the category of topological abelian groups, the surjective morphism $\alpha|C: C \to \ker\beta \cap \operatorname{im}\alpha$, where $C = \ker(\beta\alpha)$, splits as a morphism of topological groups. ∎

LEMMA 4. *A direct product of two free Markov (Graev) topological abelian groups is a free Markov (respectively, Graev) topological abelian group.*

Proof. The category of topological abelian groups is complete and co-complete. It has bi-products; that is, if $A*B$ is the co-product and $A\times B$ the product, then the natural morphism $\nu_{AB}: A*B \to A\times B$ (which exists in any category with finite products and co-products) is an isomorphism. If $A = A(X)$ and $B = A(Y)$, then $A(X)*A(Y)$ is free on $X*Y$, the co-product of X and Y in the category of topological spaces (Markov) or the category of pointed topological spaces (Graev) for categorical reasons (as the left adjoint F preserves co-products). ∎

Note that $(X\times\{1\}) \cup (\{1\}\times Y)$ is (naturally homeomorphic to) the co-product $X*Y$ and is the free generating set of $FX\times FY$. This works in both categories of spaces under consideration, pointed and non-pointed.

THEOREM 5. *Let $A(X)$ be a free Markov (Graev) abelian topological group and $\rho: A(X)\to H$ any morphism of topological groups into a discrete topological group. Then* $\ker\rho$ *is a Markov (respectively, Graev) free topological abelian group.*

Proof. Set $Y = \rho(X)$ and define $f: X\to Y$ by $f(x) = \rho(x)$. Since H is discrete, Y and $A(Y)$ are discrete. We have $\rho = \beta\circ(\bar{f})$ with a unique $\beta: A(Y)\to H$. By Proposition 2, the group $\ker(\bar{f})$ is a free topological abelian group. By Lemma 3, the group $\ker\rho$ is a direct product of $\ker(\bar{f})$ and a free discrete abelian group. The latter is trivially a free topological abelian group. In view of the foregoing, Lemma 4 shows that $\ker f$ is a free topological abelian group. ∎

COROLLARY 6. *Every open subgroup of a free Markov (Graev) abelian topological group is a free Markov (respectively, Graev) abelian topological group.*

Proof. Suppose that K is an open subgroup of $A(X)$. Then FX/K is a discrete subgroup, and K is the kernel of the quotient morphism $\rho: A(X)\to FX/K$. Theorem 4 applies to prove the assertion. ∎

Remark 1. It should be noted that the Axiom of Choice is not used in the proof of Proposition 1, but the proof of Proposition 2 and Lemma 3 requires the Axiom of Choice twice owing to the use of Schreier's Theorem and the use of the fact that free discrete groups are projective in the category of topological groups. ∎

Remark 2. It follows immediately from the basic properties of the Lebesgue covering dimension that the dimension of X and the dimension of a free topological basis for any open subgroup of $A(X)$ are the same. ∎

4. *Conclusion*

If X and Y are free topological bases of a free (abelian) topological group then $\dim X = \dim Y$. (Graev proved this for X and Y compact [**3**], and Pestov for X and Y completely regular [**13**]; for further developments, see Gul'ko [**4**].) This result bears a striking similarity to the well-known property of free bases of a discrete free (abelian) group: every two of them have the same cardinality, called the rank of the free (abelian) group. It seems natural to refer to the dimension of any free topological basis of a free (abelian) topological group as its topological rank. The rank of a subgroup of a free group can exceed the rank of the group itself, and the same is true for topological rank [**12**]. At the same time, the rank of a subgroup of a free abelian group cannot exceed the rank of the group itself, and it was conjectured [**6, 11**] that a similar statement holds for the topological rank. In our forthcoming paper [**8**] we shall refute this conjecture. Nevertheless, our present result implies that the topological rank of an *open* subgroup of a free abelian topological group is the same as the topological rank of the group itself.

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