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Generators on the arc component of compact connected groups

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Introduction

It is well-known that a compact connected abelian group G has weight $w(G)$ less than or equal to the cardinality $\mathfrak{c}$ of the continuum if and only if it is monothetic; that is, if and only if it can be topologically generated by one element. Hofmann and Morris [**2**] extended this by showing that a compact connected (not necessarily abelian) group can be topologically generated by two elements if and only if $w(G) \leqslant \mathfrak{c}$.

In any compact connected group G, the arc component of the identity plays a special role, since it is the union of the one-parameter subgroups of G. The second author asked whether it is always possible to choose a minimal set of topological generators of G from within the arc component of G. We shall prove here that this is possible.

In Hofmann and Morris [**2**] it is shown that for $w(G) > \mathfrak{c}$, the compact connected group G is not topologically generated by any finite set. In this case we look for topological generating sets which are, in some sense, 'thin'. A subset X of G is called suitable if it topologically generates G, is discrete and is closed in $G\backslash\{1\}$, where 1 is the identity of G. If X has the smallest cardinality of any suitable subset of G then G is called a special subset and its cardinality is denoted by $s(G)$. In [**2**] it was proved that if G is a connected locally compact group with $w(G) > \mathfrak{c}$, then $s(G)^{\aleph_0} = w(G)^{\aleph_0}$. It is proved here that if G is a compact connected group, then the arc component of G contains a special subset of G. As a corollary of this we deduce that the arc component of a connected locally compact group G with $w(G) > \mathfrak{c}$ contains a special subset of G.

The principal result

If G is a topological group and X is a subset we shall write $\langle\langle X\rangle\rangle$ for the smallest closed subgroup containing X.

We recall some definitions from [**2**].

Definition 1. (i) A subset X of a topological group G is called suitable if it is discrete, contained and closed in $G\backslash\{1\}$, and $G = \langle\langle X\rangle\rangle$.

(ii) If G contains suitable subsets, then we set

$$s(G) = \min\{\operatorname{card} X : X \text{ is a suitable subset of } G\}$$

and call this cardinal the generating rank of G.

(iii) A subset X of G is called special if it is suitable and card $X = s(G)$. ∎

We showed in [2] (theorem 1·12) that all locally compact groups contain suitable sets. In particular, for all locally compact groups, the generating rank is defined. Note that $s(G) \leqslant w(G)$ always. In [3] (corollary 2·16), for each infinite cardinal $\aleph_\nu$ with $\aleph_\nu < \aleph_\nu^{\aleph_0}$ we gave an example of a compact connected group G_ν such that $s(G) \leqslant \aleph_\nu$ and $w(G) = \aleph_\nu^{\aleph_0}$. In [2] (theorems 4·13 and 4·14) we proved that for a compact connected group G with $w(G) \leqslant \mathfrak{c}$ we have

$$s(G) = \begin{cases} 0 & \text{if } G \text{ is singleton,} \\ 1 & \text{if } G \text{ is abelian and non-singleton,} \\ 2 & \text{if } G \text{ is non-abelian.} \end{cases}$$

If $w(G) > \mathfrak{c}$ then $s(G)^{\aleph_0} = w(G)^{\aleph_0}$.

The arc component of the identity of G will be denoted G_a. The principal result of this paper is the following:

THEOREM 2. *Let G be a compact connected group. Then there is a special subset X of G which is contained in G_a.*

Several lemmas

The proof of Theorem 2 will proceed through several reductions. Until further notice, G will always denote a compact connected group.

LEMMA 3. *Assume that Theorem 2 is true for all abelian groups G. Then Theorem 2 is true in general.*

Proof. Let G be a compact connected non-abelian group and T a maximal protorus. (See proposition 2·4 of [2], where a maximal protorus is defined to be a maximal connected abelian subgroup of G and shown always to exist.) By hypothesis, we can find a special subset X in T_a. By corollary 2·5 of [2], there is a $g \in G$ such that $G = \langle\langle X \cup \{g\} \rangle\rangle$. Since G is the union of the conjugates of T (see [2], proposition 2·4 (ii)), there is an $h \in G$ such that $g \in hTh^{-1}$. Clearly G is topologically generated by $T \cup hTh^{-1}$. Hence $G = \langle\langle Y \rangle\rangle$ with $Y = X \cup hXh^{-1}$. Since X satisfies (i) and (iii) of Definition 1, the same is true for Y. Also, $Y \subseteq T_a \cup hT_a h^{-1} \subseteq G_a$. If $\aleph_0 \leqslant w(G) \leqslant \mathfrak{c}$, then card $X = 1$ and thus, since G is not abelian, card $Y = 2 = s(G)$ and so Y is special. If $\mathfrak{c} < w(G)$, then card Y = card $X = s(G)$, and hence Y is special. This completes the proof of the Lemma. ∎

After Lemma 3 the task is reduced to the abelian case.

LEMMA 4. *Theorem 2 is true for all abelian G with $w(G) \leqslant \mathfrak{c}$.*

Proof. By Lemma 3, is suffices to show that each connected monothetic G has a generator in G_a. Now the hypothesis that G is connected monothetic means that $\hat{G}$

is torsion free and of rank $\leqslant \mathfrak{c}$. Let $\mathbb{T}$ denote $\mathbb{R}/\mathbb{Z}$ and $p:\mathbb{R}\to\mathbb{T}$ the quotient homomorphism. The group $\mathbb{T}$ is algebraically isomorphic to $\mathbb{Q}/\mathbb{Z}\oplus\mathbb{R}$. Hence there is an injective morphism, $j:\hat{G}\to\mathbb{R}$ such that $p\circ j:\hat{G}\to\mathbb{T}$ remains injective. Hence the dual $\hat{j}\circ\hat{p}:\mathbb{Z}\to G=\hat{\hat{G}}$ has dense image and factors through $\hat{p}:\mathbb{Z}\to\mathbb{R}$. Thus $\hat{j}\hat{p}(1)$ is a generator on the arc component of the identity. ∎

Now the only remaining case is: G is abelian and $w(G)>\mathfrak{c}$; that is, $\hat{G}$ is torsion free of rank $>\mathfrak{c}$.

Lemma 5. *Let X be a suitable subset of a topological group H such that $X\cup\{1\}$ is compact. Assume that $f:H\to K$ is a morphism of topological groups with dense image. Then $f(X)\backslash\{1\}$ is a suitable subset of K. If X is special and if $s(H)\leqslant s(K)$ then $f(X)\backslash\{1\}$ is special.*

Proof. Since X is discrete and closed in $G\backslash\{1\}$, and since $X\cup\{1\}$ is compact, then for every identity neighbourhood U in H the set $X\backslash U$ is finite. Now assume that V is an identity neighbourhood of K. Then $X\backslash f^{-1}(V)$ is finite. Since $h\in f(X)\backslash V$ implies $h=f(x)$ with $x\in X\backslash f^{-1}(V)$, then $f(X)\backslash V$ is finite. Thus $f(X)\backslash\{1\}$ is discrete and $f(X\cup\{1\})=f(X)\cup\{1\}$ is compact. Hence $f(X)\backslash\{1\}$ is closed. Also

$$K=\overline{f(H)}=\overline{f(\langle\!\langle X\rangle\!\rangle)}\subseteq\overline{\langle\!\langle f(X)\rangle\!\rangle}=\langle\!\langle f(X)\rangle\!\rangle.$$

So $f(X)\backslash\{1\}$ is a suitable subset of K. Finally, $\operatorname{card}(f(X)\backslash\{1\})\leqslant\operatorname{card}X=s(H)$, whence $s(K)\leqslant\operatorname{card}(f(X)\backslash\{1\})\leqslant s(H)$. Thus $f(X)\backslash\{1\}$ is special if $s(H)\leqslant s(K)$. ∎

Before we proceed with the next lemma we recall that each locally compact abelian group H has an exponential function $\exp:L(H)\to H$ such that $L(G)=\operatorname{Hom}(\mathbb{R},G)$, $\exp X=X(1)$, and $G_a=\exp L(G)$. (For further comments, see [1], remark 2·2·2.) We give $L(G)$ the topology of uniform convergence on compact sets. Note that we have an isomorphism $\alpha:L(G)\to\operatorname{Hom}(\hat{G},\mathbb{R})$, $\alpha(f)=\hat{f}$ (setting $\hat{\mathbb{R}}=\mathbb{R}$ with the pairing $(r,s)\mapsto rs:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$) and an isomorphism $\beta:G\to\operatorname{Hom}(G,\mathbb{T})$, $\beta(g)(\chi)=\chi(g)$. Here $\operatorname{Hom}(\hat{G},\mathbb{R})$ and $\operatorname{Hom}(\hat{G},\mathbb{T})$ both have the topology of pointwise convergence. There is a commutative diagram

$$\begin{array}{ccc} L(G) & \xrightarrow{\ \exp\ } & G \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \operatorname{Hom}(\hat{G},\mathbb{R}) & \xrightarrow[\operatorname{Hom}(\hat{G},p)]{} & \operatorname{Hom}(\hat{G},\mathbb{T}). \end{array}$$

Proposition 6. *Let G be a compact connected abelian group with $w(G)>\mathfrak{c}$. There is a suitable subset Y of $L(G)$ with $Y\cup\{0\}$ compact and $s(L(G))\leqslant\operatorname{card}Y=s(G)$.*

Before we prove Proposition 6 in several steps, we observe, that Proposition 6 will finish the proof of Theorem 2, the main result: indeed, if Y is a suitable subset of $L(G)$, the fact that the exponential function is a morphism with dense image, by Lemma 5, implies $\exp Y$ is a suitable subset of G contained in $G_a=\exp L(G)$. This is what we claim in Theorem 2.

The proof of Proposition 6 requires several further lemmas. The first of these is proved by diagram chasing.

LEMMA 7 (Diagram Lemma). *Consider the commutative diagram of abelian groups with exact columns. If the first two rows are exact, then the third row is exact.*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \xrightarrow{\alpha_1} & B_1 & \xrightarrow{\beta_1} & C_1 & \longrightarrow & 0\\
 & & {\scriptstyle\pi_1}\downarrow & & {\scriptstyle\rho_1}\downarrow & & \downarrow{\scriptstyle\sigma_1} & & \\
0 & \longrightarrow & A_2 & \xrightarrow{\alpha_2} & B_2 & \xrightarrow{\beta_2} & C_2 & \longrightarrow & 0\\
 & & {\scriptstyle\pi_2}\downarrow & & {\scriptstyle\rho_2}\downarrow & & \downarrow{\scriptstyle\sigma_2} & & \\
0 & \longrightarrow & A_3 & \xrightarrow{\alpha_3} & B_3 & \xrightarrow{\beta_3} & C_3 & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

If X is a pointed compact space and $\mathbb{K} \in \{\mathbb{Z}, \mathbb{R}, \mathbb{T}, \mathbb{Q}, \mathbb{Q}/\mathbb{Z}\}$, we shall write $C(X, \mathbb{K})$ for the abelian group of all base-point preserving continuous functions under pointwise addition. Further, if A is a subgroup of $\mathbb{K}$, then $C_{\mathrm{fin}}(X, A)$ will denote the subgroup of $C(X, \mathbb{K})$ consisting of all functions taking only finitely many values in A. Finally, $[X, \mathbb{T}]$ is the group of all homotopy classes of continuous base-point preserving functions $X \to \mathbb{T}$. We recall that $[X, \mathbb{T}] \cong H^1(X, \mathbb{Z})$ (see [**1**]).

LEMMA 8. *For a compact pointed space X such that $[X, \mathbb{T}] = 0$ we have*

$$C(X, \mathbb{R})/C_{\mathrm{fin}}(X, \mathbb{Q}) \cong C(X, \mathbb{T})/C_{\mathrm{fin}}(X, \mathbb{Q}/\mathbb{Z}).$$

Proof. The exact sequence

$$0 \to \mathbb{Z} \overset{j}{\to} \mathbb{R} \overset{p}{\to} \mathbb{T} \to 0$$

induces an exact sequence

$$0 \to C(X, \mathbb{Z}) \overset{j^*}{\to} C(X, \mathbb{R}) \overset{p^*}{\to} C(X, \mathbb{T}) \to [X, \mathbb{T}] \to 0$$

(see [**1**]). We now assume that $[X, \mathbb{T}] = \{0\}$. We set $B^* = C(X, \mathbb{R})/C_{\mathrm{fin}}(X, \mathbb{Q})$ and $B = C(X, \mathbb{T})/C_{\mathrm{fin}}(X, \mathbb{Q}/\mathbb{Z})$. Then we have a commutative diagram with exact columns whose first two rows are exact:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{\mathrm{fin}}(X,\mathbb{Z}) & \longrightarrow & C_{\mathrm{fin}}(X,\mathbb{Q}) & \longrightarrow & C_{\mathrm{fin}}(X,\mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C(X,\mathbb{Z}) & \longrightarrow & C(X,\mathbb{R}) & \longrightarrow & C(X,\mathbb{T}) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & B^* & \longrightarrow & B & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

By the Diagram Lemma 7, the assertion follows. ∎

LEMMA 9. *Let X denote a compact space. Then, as rational vector spaces,* $C(X, \mathbb{R}) \cong C(X, \mathbb{R})/C_{\text{fin}}(X, \mathbb{Q})$.

Proof. Write $R = \mathbb{Q} \oplus E$ with a suitable $\mathbb{Q}$-vector space complement E for $\mathbb{Q}$ in $\mathbb{R}$. Then $C_{\text{fin}}(X, \mathbb{Q}) \cap C_{\text{fin}}(X, E) = \{0\}$ and thus there is a vector space complement $\mathscr{F}$ of $C_{\text{fin}}(X, \mathbb{Q})$ in $C(X, \mathbb{R})$ containing $C_{\text{fin}}(X, E)$. We note that $E \cong \mathbb{Q}^{(\mathfrak{c})}$ and thus $C_{\text{fin}}(X, E) \cong C_{\text{fin}}(X, \mathbb{Q})^{(\mathfrak{c})}$, and $\mathscr{F}$ contains a vector subspace $\mathscr{V} \cong C_{\text{fin}}(X, \mathbb{Q})^{(\mathfrak{c})}$. We write $\mathscr{F} = \mathscr{V} \oplus \mathscr{W}$. Therefore

$$C(X, \mathbb{R}) \cong C_{\text{fin}}(X, \mathbb{Q}) \oplus \mathscr{F} = C_{\text{fin}}(X, \mathbb{Q}) \oplus \mathscr{V} \oplus \mathscr{W} \cong \mathscr{V} \oplus \mathscr{W} = \mathscr{F}.$$

Since $\mathscr{F} \cong C(X, \mathbb{R})/C_{\text{fin}}(X, \mathbb{Q})$ the assertion follows. ∎

LEMMA 10. (i) *If X is a compact pointed space such that $\dim_{\mathbb{R}} C(X, \mathbb{R}) \geqslant \mathfrak{c}$ then, for every subgroup A of $C(X, \mathbb{R})$, there is an injective $\mathbb{R}$-linear map $\mathbb{R} \otimes_{\mathbb{Z}} A \to C(X, \mathbb{R})$.*

(*ii*) *If X is a compact pointed space with $w(X) > \mathfrak{c}$ then $\dim_{\mathbb{R}} C(X, \mathbb{R}) > \mathfrak{c}$ and so Part* (i) *applies.*

Proof. (i) The inclusion $j: A \to C(X, \mathbb{R})$ induces an injective $\mathbb{R}$ linear map $\text{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} j: \mathbb{R} \otimes_{\mathbb{Z}} A \to \mathbb{R} \otimes_{\mathbb{Z}} C(X, \mathbb{R})$ because $\mathbb{R}$ is torsion-free. The assertion will be proved if we show that the $\mathbb{R}$-vector spaces $\mathbb{R} \otimes_{\mathbb{Z}} C(X, \mathbb{R})$ and $C(X, \mathbb{R})$ are isomorphic. For this it suffices to show that their $\mathbb{R}$-dimensions are equal.

Let S denote a set. Then, as $\mathbb{Q}$-vector spaces, $\mathbb{R}^{(S)} \cong (\mathbb{Q}^{(\mathfrak{c})})^{(S)} \cong \mathbb{Q}^{(\mathfrak{c}.S)}$. Thus $\text{card}\, \mathbb{R}^{(S)} = \mathfrak{c}.\text{card}\, S$. If V is a real vector space, then

$$\text{card}\, V = \mathfrak{c}.\dim_{\mathbb{R}} V \tag{$*$}$$

and if $\dim_{\mathbb{R}} V \geqslant \mathfrak{c}$, then $\dim_{\mathbb{R}} V = \text{card}\, V$.

Now

$$\mathbb{R} \otimes_{\mathbb{Z}} C(X, \mathbb{R}) \cong \mathbb{R}^{(\dim_{\mathbb{Q}} C(X, \mathbb{R}))} = \mathbb{R}^{(\text{card}\, C(X, \mathbb{R}))}$$

because $\dim_{\mathbb{Q}} C(X, \mathbb{R})$ is infinite. Further, card $C(X, \mathbb{R}) = w(X)^{\aleph_0}$ (see [1] and errata). Thus $\mathbb{R} \otimes_{\mathbb{Z}} C(X, \mathbb{R}) \cong \mathbb{R}^{(w(X)^{\aleph_0})}$. Hence $\dim_{\mathbb{R}} \mathbb{R} \otimes_{\mathbb{Z}} C(X, \mathbb{R}) = w(X)^{\aleph_0}$ and $\text{card}\, C(X, \mathbb{R}) = w(X)^{\aleph_0}$. Since $\dim_{\mathbb{R}} C(X, \mathbb{R})$ was assumed to be at least $\mathfrak{c}$ we conclude

$$\dim_{\mathbb{R}} C(X, \mathbb{R}) = w(X)^{\aleph_0}.$$

This gives the desired equality of dimensions.

(ii) For infinite X we know $\text{card}\, C(X, \mathbb{R}) = w(X)^{\aleph_0}$. Thus $w(X) > \mathfrak{c}$ implies $\text{card}\, C(X, \mathbb{R}) > \mathfrak{c}$. If $\dim C(X, \mathbb{R}) \leqslant \mathfrak{c}$, then

$$\text{card}\, C(X, \mathbb{R}) = \mathfrak{c}.\dim_{\mathbb{R}} C(X, \mathbb{R}) \leqslant \mathfrak{c}.$$

Therefore $\dim_{\mathbb{R}} C(X, \mathbb{R}) > \mathfrak{c}$, as asserted. ∎

LEMMA 11. *Let A denote an abelian torsion group, B a torsion-free abelian group and C a torsion-free subgroup of $A \oplus B$. Then the projection $p: A \oplus B \to B$ maps C injectively into B.*

Proof. Since $\ker p = A$ we have $\ker (p|C) = A \cap C$. As A is a torsion group and C is torsion-free we have $A \cap C = \{0\}$. Thus $p|C$ is injective. ∎

LEMMA 12. *Let A be a subgroup of $C(X, \mathbb{T})$ for a compact space X with $w(X) > \mathfrak{c}$ and with $[X, \mathbb{T}] = 0$. Then there is an injective linear map $\mathbb{R} \otimes_{\mathbb{Z}} A \to C(X, \mathbb{R})$.*

Proof. Since $[X, \mathbb{T}] = 0$ the group $C(X, \mathbb{T})$ is a quotient of $C(X, \mathbb{R})$ and thus is divisible. Hence its torsion subgroup $C_{\text{fin}}(X, \mathbb{Q}/\mathbb{Z})$ is a direct summand. Thus Lemma 11 applies and shows that A is isomorphic to a subgroup of $C(X, \mathbb{T})/C_{\text{fin}}(X, \mathbb{Q}/\mathbb{Z})$. This latter group is isomorphic to $C(X, \mathbb{R})$ by Lemmas 8 and 9. Thus A is isomorphic to a subgroup of $C(X, \mathbb{R})$. But then Lemma 10 applies and proves the claim. ∎

LEMMA 13. *Let E be a real topological vector space and X a subset of E such that E is the closed linear span of X. Then, as an additive topological group, $E = \langle\langle X \cup \sqrt{2}X \rangle\rangle$. If X is discrete and closed in $E\backslash\{0\}$, then $X \cup \sqrt{2}X$ is suitable.*

Proof. For each $x \in X$, the group $\langle\langle X \cup \sqrt{2}X \rangle\rangle$ contains $\mathbb{R}.x = \langle\langle \mathbb{Z} + \sqrt{2}\mathbb{Z} \rangle\rangle.x$, hence it contains the linear span of X.

If X is discrete, then $X \cup \sqrt{2}X$ is discrete, and if X is closed in $E\backslash\{0\}$ then so is $X \cup \sqrt{2}X$. ∎

LEMMA 14. *Let V be a real vector space and V^* the algebraic dual with the topology of pointwise convergence. Denote by $(V^*)'$ the topological dual of V^*. Then $e: V \to (V^*)'$, $e(v)(\alpha) = a(v)$ is an isomorphism of $\mathbb{R}$-vector spaces.*

Proof. Since V^* separates the points of V, clearly e is injective. Let $\Omega: V^* \to \mathbb{R}$ be a continuous linear functional. Let $U = \Omega^{-1}(]-1, 1[)$. Then by the definition of the topology of pointwise convergence on V^*, there are vectors $v_1, \ldots, v_n \in V$ and there is an $\epsilon > 0$ such that $|\alpha(v_j)| < \epsilon$, $j = 1, \ldots, n$ implies $\alpha \in U$; that is, $|\Omega(\alpha)| < 1$. Let F denote the span of the v_j and $A = F^\perp$ the vector space of all $\alpha \in V^*$ vanishing on all v_j. then $\Omega(A)$ is a vector subspace of $]-1, 1[$ and is, therefore $\{0\}$. Thus Ω induces a linear functional ω on $V^*/A \cong F^*$; that is, $\omega \in F^{**}$. Hence, by the duality of finite-dimensional vector spaces, there is a $w \in F$ such that $\omega(\alpha + A) = \alpha(w)$. It follows that $\Omega(\alpha) = \alpha(w)$ and thus $\Omega = e(w)$. Thus e is surjective, too. ∎

LEMMA 15. *The closed $\mathbb{R}$-linear span of $\eta(X')$ is $C(X', \mathbb{R})$.*

Proof. Set $E = \langle\langle \mathbb{R}.\eta(X') \rangle\rangle$, the closed $\mathbb{R}$-linear span of $\eta(X')$ in $C(X', \mathbb{R})^*$. We claim that $E = C(X', \mathbb{R})^*$. If not, then there is a non-zero continuous linear functional $\Omega: C(X', \mathbb{R})^* \to \mathbb{R}$ vanishing on E by the Hahn–Banach Theorem. Now we apply Lemma 14 with $V = C(X', \mathbb{R})$ and find that there is an $f \in C(X', \mathbb{R})$ such that $\Omega(\alpha) = a(f)$. Hence $E(f) = \{0\}$. In particular, $f(x) = \eta(x)(f) = 0$ for all $x \in X'$. Thus $f = 0$ and therefore $\Omega = 0$, a contradiction. Thus $E = C(X', \mathbb{R})$ is proved. ∎

Now we are ready for a proof of Proposition 6. Thus we consider a compact connected abelian group G with weight $w(G) > \mathfrak{c}$. We know that $s(G)^{\aleph_0} = w(G)^{\aleph_0}$. If we had $s(G) \leqslant \mathfrak{c}$, then

$$w(G) \leqslant w(G)^{\aleph_0} = s(G)^{\aleph_0} \leqslant \mathfrak{c}^{\aleph_0} = \mathfrak{c},$$

in contradiction to our hypothesis. Thus G contains a special subset X of cardinality $s(G) > \mathfrak{c}$ such that $X' = X \cup \{1\}$ is compact. For infinite suitable sets X we have $w(X') = \operatorname{card} X$. Thus $w(X') > \mathfrak{c}$. Since the pointed space X' is generating, the natural morphism $f: FX' \to G$ from the free compact abelian group FX' on X' to G satisfying $f(x) = x$ for $x \in X$ is surjective. Hence $\hat{f}: \hat{G} \to \widehat{FX'}$ is injective. But $\widehat{FX'} \cong C(X', \mathbb{T})$ (see [1]). By Lemma 12 we thus have an injective $\mathbb{R}$-linear map $j: \mathbb{R} \otimes_{\mathbb{Z}} \hat{G} \to C(X', \mathbb{R})$. Its dual $\operatorname{Hom}_{\mathbb{R}}(j, \mathbb{R}): \operatorname{Hom}_{\mathbb{R}}(C(X', \mathbb{R}), \mathbb{R}) \to \operatorname{Hom}(\mathbb{R} \otimes_{\mathbb{Z}} \hat{G}, \mathbb{R})$ is a surjective continuous $\mathbb{R}$-linear map between topological vector spaces. But

$$\operatorname{Hom}_R(\mathbb{R} \otimes_{\mathbb{Z}} \hat{G}, \mathbb{R}) \cong \operatorname{Hom}(\hat{G}, \mathbb{R}) \cong L(G).$$

Thus we have produced a continuous surjective $\mathbb{R}$-vector space morphism $j^*: C(X', \mathbb{R})^* \to L(G)$, where E^* denotes the algebraic dual of a real vector space E endowed with the topology of pointwise convergence. The natural map $\eta: X' \to C(X', \mathbb{R})^*$, $\eta(x)(f) = f(x)$ is a topological embedding since the continuous functions on a compact space separate the points, since the topology of $C(X', \mathbb{R})^*$ is that of pointwise convergence, and since X' is compact. By Lemma 5 we know that $Z = j^*(\eta(X')\backslash\{0\})$ is discrete and closed in $L(G)\backslash\{0\}$ and is such that $Z \cup \{0\}$ is compact. By Lemma 15, the closed $\mathbb{R}$-linear span of $\eta(X')$ is $C(X', \mathbb{R})$. Hence the closed $\mathbb{R}$-linear span of Z is $L(G)$. Then by Lemma 13, the set $Y = Z \cup \sqrt{2}Z$ is suitable in $L(G)$. By Lemma 5 we know that $\exp Y$ is suitable in G. Hence

$$s(G) \leqslant \operatorname{card}(\exp Y) \leqslant \operatorname{card} Y \leqslant \operatorname{card} X = s(G).$$

So $\exp Y$ is a special subset of G. Since Y is a suitable subset of $L(G)$ we have $s(L(G)) \leqslant \operatorname{card} Y = s(G)$.

This completes the proofs of Proposition 6 and of Theorem 2. ∎

We do not know whether in fact Y is special in $L(G)$ and $s(L(G)) = s(G)$. This is left as an open question.

Some consequences

We shall draw some conclusions on the locally compact case.

LEMMA 16. *Let G be a locally compact connected group. Then there is a compact normal subgroup N and a connected Lie group L and an injective morphism $\Phi: L \to G$ such that* (i) $[N, \phi(L)] = \{1\}$, (ii) $G = N\phi(L)$, *and* (iii) *there is an identity neighbourhood U in L such that $(n, u) \mapsto n\phi(u): N \times U \to N\phi(U)$ is a homeomorphism onto an identity neighbourhood of* 1 *such that $[N, \phi(U)] = \{1\}$.*

Proof. (i) and (ii) are consequences of (iii), and (iii) is Iwasawa's local product theorem (see [**5**]).

LEMMA 17. *Let everything be as in Lemma* 16. *Then $G_a = N_a\phi(L)$.*

Proof. Since the subgroup $N_a\phi(L)$ is arc-connected we have $N_a\phi(L) \subseteq G_a$ and we now must prove the reverse containment. We shall do this by showing that for every one-parameter subgroup $X: \mathbb{R} \to G$ of G we have $X(\mathbb{R}) \subseteq N_a\phi(L)$. This will suffice since

G_a is generated by all one-parameter subgroups.

Set $f: N \times L \to G$, $f(n, g) = n\phi(g)$. Then f is a surjective morphism of a σ-compact locally compact group onto a locally compact group. Hence it is open. Now by the lifting theorem for one parameter groups there is a one parameter group $Y: \mathbb{R} \to N \times L$ such that $X = f \circ Y$. (See e.g. [**4**], lemma 1·3.) Now there are one-parameter groups $Y_1: \mathbb{R} \to N$ and $Y_2: \mathbb{R} \to L$ such that $Y(r) = (Y_1(r), Y_2(r))$ for all $r \in \mathbb{R}$. Then $Y_1(\mathbb{R}) \subseteq N_a$. Hence $X(r) = f((Y_1(r), Y_2(r))) \subseteq Y_1(\mathbb{R})\phi(Y_2(\mathbb{R})) \subseteq N_a\phi(L)$. ∎

The principal result on compact connected groups, Theorem 2, has the following corollary:

maps onto G under a quotient homomorphism, we have $w(N\times L) \geqslant w(G) > \mathfrak{c}$. Since $w(L) = \aleph_0$ we have $w(N) > \mathfrak{c}$. The Lie group L has a finite topological generating set F. We find a special subset S of N inside N_a by Theorem 2. Also, $X = S \cup \phi(F)$ is a suitable subset of G whose cardinality is $\operatorname{card} S = s(N)$. Now $\overline{\phi(L)}$ is a compact connected normal subgroup H of G with weight $w(H) \leqslant \mathfrak{c}$. Now $N/(N\cap H) \cong G/H$. Thus $s(G/H) \leqslant s(N)$. Hence $s(G) \leqslant s(G/H)+s(H) \leqslant s(N)+w(H) \leqslant s(N)+\mathfrak{c} = s(N)$ since $s(N)^{\aleph_0} = w(X)^{\aleph_0} \geqslant w(X) > \mathfrak{c}$ and thus $s(N) > \mathfrak{c}$. Thus the cardinal of X is $s(G)$ and thus X is special. ∎

The methods used in the proof of this corollary allow us to conclude also that every locally compact connected group G has a suitable subset of G in G_a. But it is not immediate whether a special subset of G can be found inside G_a if $w(G) \leqslant \mathfrak{c}$ and thus $s(G)$ is finite. This is the topic of another investigation.

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