

GENERATORS FOR LOCALLY COMPACT GROUPS

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It is proved that if G is any compact connected Hausdorff group with weight $w(G) \leq c$, \mathbb{R} is the topological group of all real numbers and n is a positive integer, then the topological group $G \times \mathbb{R}^n$ can be topologically generated by $n+1$ elements, and no fewer elements will suffice.

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1. Introduction

A locally compact group is said to be *monothetic* if it can be topologically generated by one element. Within the class of compact connected Hausdorff abelian groups, the monothetic groups are those which have weight at most c , the cardinality of the continuum [4, §25.14]. Hofmann and Morris [5, Theorem 4.13] have shown that a compact connected Hausdorff group G is topologically generated by two elements if and only if G has weight $w(G) \leq c$.

Kronecker's approximation theorem (for example, [6, Theorem 28]) can be interpreted as saying that the number of elements required to topologically generate \mathbb{R}^n is exactly $n+1$, where n is any non-negative integer.

In this paper we show that if G is a compact connected Hausdorff group with weight $w(G) \leq c$, and n is a positive integer, then the number of elements required to topologically generate $G \times \mathbb{R}^n$ is exactly $n+1$. Note that the class of topological groups of this form is just the class of connected locally compact maximally almost periodic groups of weight at most c [3].

1.2. Definitions and notation

- (i) A topological group G is said to be *topologically finitely generated* if there exists a positive integer m such that $G = \text{gp}\{x_1, x_2, \dots, x_m\}$, for some set $\{x_1, x_2, \dots, x_m\} \subseteq G$. If n is the smallest such integer, then we say $\sigma(G) = n$.
- (ii) Let $\{K_j: j \in J\}$ be the family of all compact connected simple simply-connected Lie groups. For each $j \in J$, let $C_j = (K_j)^c$, the product of c copies of K_j , where c is the cardinality of the continuum. Then we define

$$\mathcal{G} = \prod_{j \in J} C_j$$

As each K_j is a simple Lie group, $(K_j)' = K_j$ [2, 5 Corollaire de la Proposition 10]. So it follows that \mathcal{C} is its own commutator subgroup; that is, $\overline{\mathcal{C}'} = \mathcal{C}' = \mathcal{C}$.

2. Preliminary results

The first five results are folklore, but we include some proofs for completeness.

Lemma 2.1. *Let A be a closed normal subgroup of a topological group B and let C be the quotient group B/A . If A and C are topologically finitely generated, then B is topologically finitely generated and $\sigma(B) \leq \sigma(A) + \sigma(C)$. □*

Lemma 2.2. *Let G and H be topological groups. If G is topologically finitely generated and there exists a continuous homomorphism of G onto H , then H is topologically finitely generated and $\sigma(G) \geq \sigma(H)$. □*

The following proposition can be considered an interpretation of Kronecker's approximation theorem, [6, Theorem 28].

Proposition 2.3. *If n is a positive integer, then $\sigma(\mathbb{R}^n) = n + 1$.*

Proof. We have the following exact sequence:

$$0 \rightarrow \mathbb{Z}^n \rightarrow \mathbb{R}^n \rightarrow \mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n \rightarrow 0.$$

But $\sigma(\mathbb{Z}^n) = n$ and $\sigma(\mathbb{T}^n) = 1$ [4, § 25.14], and so by Lemma 2.1, $\sigma(\mathbb{R}^n) \leq n + 1$.

Now suppose that $\sigma(\mathbb{R}^n) = m$ for some $m \leq n$. Then there exists a set $B = \{b_1, b_2, \dots, b_m\} \subseteq \mathbb{R}^n$ such that $\mathbb{R}^n = \overline{\text{gp}\{B\}}$.

If B is a linearly dependent set and $\text{sp}\{B\}$ the vector subspace spanned by B , then

$$\begin{aligned} \overline{\text{gp}\{B\}} &\subseteq \overline{\text{sp}\{B\}} \\ &= \overline{\text{sp}\{b_1, b_2, \dots, b_k\}}, \text{ for some } k < m \leq n \\ &= \mathbb{R}^k, \text{ as all vector subspaces of } \mathbb{R}^n \text{ are closed.} \end{aligned}$$

Therefore $\overline{\text{gp}\{B\}} \neq \mathbb{R}^n$.

If B is linearly independent set, then, from Proposition 21 of [6], $\overline{\text{gp}\{B\}}$ is topologically isomorphic to \mathbb{Z}^m , and again we have $\overline{\text{gp}\{B\}} \neq \mathbb{R}^n$.

Therefore $\sigma(\mathbb{R}^n) \neq m$ for any $m \leq n$, and so $\sigma(\mathbb{R}^n) = n + 1$. □

Proposition 2.4. *Let K be a compact connected Hausdorff abelian group of weight $w(K) \leq c$, and n a non-negative integer. Then $\sigma(K \times \mathbb{R}^n) = n + 1$.*

Proof. Consider the exact sequence

$$0 \rightarrow \mathbb{Z}^n \rightarrow K \times \mathbb{R}^n \rightarrow K \times \mathbb{T}^n \rightarrow 0.$$

As $K \times \mathbb{T}^n$ is a compact connected Hausdorff abelian group with weight $w(K \times \mathbb{T}^n) \leq c$, [4, §25.14] implies that $\sigma(K \times \mathbb{T}^n) = 1$. So by Lemma 2.1, we obtain $K \times \mathbb{R}^n$ is topologically finitely generated and $\sigma(K \times \mathbb{R}^n) \leq \sigma(\mathbb{Z}^n) + \sigma(K \times \mathbb{T}^n)$. But, $\sigma(\mathbb{Z}^n) = n$ and so $\sigma(K \times \mathbb{R}^n) \leq n + 1$.

But as \mathbb{R}^n is a quotient of $K \times \mathbb{R}^n$, $\sigma(K \times \mathbb{R}^n) \geq \sigma(\mathbb{R}^n) = n + 1$, from Proposition 2.3. Hence $\sigma(K \times \mathbb{R}^n) = n + 1$. □

Lemma 2.5. ([1, §9.1.1]) *Let A and B be subsets of a topological group G . Then $[A, B] = \overline{[A, B]}$.* □

Theorem 2.6. ([5, Theorem 4.13]) *If G is a compact connected Hausdorff non-abelian topological group of weight $w(G) \leq c$, then $\sigma(G) = 2$.* □

Lemma 2.7. *Let A and B be subgroups of a group G . Then A and B are contained in the normalizer of $[A, B]$. Further, if G is generated by $A \cup B$ and A and B are abelian, then $G' = [A, B]$.*

Proof. That the normalizer of $[A, B]$ contains A and B follows from the identity

$$[xy, z] = [y, z]^{x^{-1}}[x, z].$$

If G is generated by $A \cup B$ then it follows that $[A, B]$ is a normal subgroup of G . If A and B are abelian, then obviously $G/[A, B]$ is abelian. Hence $G' \subseteq [A, B]$. As $G' \supseteq [A, B]$ trivially, we deduce that $G' = [A, B]$. □

Proposition 2.8. *Let G be a topological group topologically generated by two elements, a and b . Then $\overline{G'} = \overline{\text{gp}\{a\}, \text{gp}\{b\}}$.*

Proof. From Lemma 2.7, we have $\overline{\text{gp}\{a\}, \text{gp}\{b\}} = \overline{(\text{gp}\{a, b\})'}$, so

$$\begin{aligned} \overline{G'} &= \overline{\overline{\text{gp}\{a, b\}, \text{gp}\{a, b\}}}, \\ &= \overline{\text{gp}\{a, b\}, \text{gp}\{a, b\}}, \text{ from Lemma 2.5} \\ &= \overline{(\text{gp}\{a, b\})'} \\ &= \overline{\text{gp}\{a\}, \text{gp}\{b\}}. \end{aligned}$$
□

3. The main result

The following result is the key to Theorem 3.2, the main result of this paper.

Theorem 3.1. *Let G be any compact connected Hausdorff group of weight $w(G) \leq c$. Then there exists a compact connected Hausdorff abelian group K of weight $w(K) \leq c$ and a continuous homomorphism from $K \times \mathcal{C}$ onto G .*

Proof. There exists a family of compact connected simple simply-connected Lie groups $\{G_i : i \in I\}$, a compact connected abelian subgroup K of G , and a continuous homomorphism ϕ of $\prod_{i \in I} G_i \times K$ onto G , with $\ker \phi$ a totally disconnected subgroup [7, Theorem 6.5.6]. Note that $w(K) \leq w(G) \leq c$.

From Proposition 2.2 of [5], $w(G) = w(\prod_{i \in I} G_i \times K)$. But, as $w(G) \leq c$, $w(\prod_{i \in I} G_i) \leq c$. So $\max\{\text{card } I, w(G_i), w(K)\} \leq c$ and therefore $\text{card } I \leq c$.

Note that the family of all compact connected simple simply-connected Lie groups is countable. It follows from this, and the fact that $\text{card } I \leq c$, that there is a continuous homomorphism of \mathcal{C} into $\prod_{i \in I} G_i$. Therefore there is a continuous homomorphism ψ of $\mathcal{C} \times K$ onto $\prod_{i \in I} G_i \times K$.

Finally, the map $\theta = \phi \circ \psi$ is the required continuous homomorphism of $\mathcal{C} \times K$ onto G . □

Theorem 3.2. *Let G be any compact connected Hausdorff group with weight $w(G) \leq c$, and n a positive integer. Then $\sigma(G \times \mathbb{R}^n) = n + 1$.*

Proof. If G is abelian, then from Proposition 2.4, $\sigma(G \times \mathbb{R}^n) = n + 1$. Now assume that G is non-abelian.

From Theorem 3.1, $G \times \mathbb{R}^n$ is a quotient group of $\mathcal{C} \times K \times \mathbb{R}^n$ for some compact connected Hausdorff abelian group K of weight $w(K) \leq c$. We will show that $\mathcal{C} \times K \times \mathbb{R}^n$ can be topologically generated by $n + 1$ elements. From this we can deduce that $G \times \mathbb{R}^n$ can be topologically generated by $n + 1$ elements. As \mathbb{R}^n is a quotient group of $G \times \mathbb{R}^n$, using Proposition 2.3 we will then see that $\sigma(G \times \mathbb{R}^n) \geq \sigma(\mathbb{R}^n) = n + 1$. This will complete the proof.

As \mathcal{C} is a compact connected Hausdorff group with $w(\mathcal{C}) \leq c$, Theorem 2.6 implies that $\sigma(\mathcal{C}) = 2$. Let $\{a, b\}$ topologically generate \mathcal{C} ,

From Proposition 2.4, we have $\sigma(K \times \mathbb{R}^n) = n + 1$. Let $\{f_1, f_2, \dots, f_{n+1}\}$ be a set of topological generators for $K \times \mathbb{R}^n$.

We shall show that the set $S = \{(a, f_1), (b, f_2), (b, f_3), \dots, (b, f_{n+1})\}$ topologically generates $\mathcal{C} \times K \times \mathbb{R}^n$.

We prove that $H = \text{gp}\{(a^{-p}b^{-q}a^pb^q, e) : p, q \in \mathbb{Z}\} \subseteq \text{gp}\{S\}$, where e is the identity of the group. To do this, it is enough to show that for any $p, q \in \mathbb{Z}$, $(a^{-p}b^{-q}a^pb^q, e) \in \text{gp}\{S\}$. Let $p, q \in \mathbb{Z}$. As K is abelian, for any $i \in \{2, 3, \dots, n + 1\}$,

$$\begin{aligned} (a^{-p}b^{-q}a^pb^q, e) &= (a^{-p}, f_1^{-p})(b^{-q}, f_i^{-q})(a^p, f_1^p)b^q, f_i^q \\ &= (a, f_1)^{-p}(b, f_i)^{-q}(a, f_1)^p(b, f_i)^q \\ &\in \text{gp}\{S\}. \end{aligned}$$

By Proposition 2.8, this implies that $\overline{\mathcal{C}' \times \{e\}} = \overline{H} \subseteq \overline{\text{gp}\{S\}}$. But, as observed in 1.2(ii), $\mathcal{C}' = \mathcal{C}$. So $\mathcal{C}' \times \{e\} \subseteq \text{gp}\{S\}$.

As $(a^{-1}, e) \in \mathcal{C} \times \{e\}$ and $(a, f_1) \in \text{gp}\{S\}$, it follows that $(a^{-1}, e)(a, f_1) = (e, f_1) \in \overline{\text{gp}\{S\}}$. Similarly, $(e, f_i) \in \overline{\text{gp}\{S\}}$ for $i \in \{2, 3, \dots, n+1\}$.

We now have $\{e\} \times (K \times \mathbb{R}^n) \subseteq \overline{\text{gp}\{S\}}$ and $\mathcal{C} \times \{e\} \subseteq \overline{\text{gp}\{S\}}$, and so $\mathcal{C} \times (K \times \mathbb{R}^n) \subseteq \overline{\text{gp}\{S\}}$. Therefore $\mathcal{C} \times (K \times \mathbb{R}^n) = \overline{\text{gp}\{S\}}$, as required.

Hence $\sigma(\mathcal{C} \times K \times \mathbb{R}^n) \leq n+1$, and the result follows. □

Corollary. *Let L be a connected locally compact Hausdorff abelian group of weight $w(L) \leq c$, and G a compact connected Hausdorff group of weight $w(G) \leq c$. If L is compact and G is abelian, then $\sigma(G \times L) = 1$. If L is compact and G is non-abelian, then $\sigma(G \times L) = 2$. If L is non-compact, then it is topologically isomorphic to $\mathbb{R}^n \times K$ for n a positive integer and K a compact connected Hausdorff group, and $\sigma(G \times L) = n+1$.*

Proof. If L is compact and G is abelian, then $G \times L$ is a compact connected Hausdorff abelian group, and so from [4, §25.14], $\sigma(G \times L) = 1$.

If L is compact and G is non-abelian, then $G \times L$ is a compact connected Hausdorff non-abelian group. Therefore, from Theorem 2.6, $\sigma(G \times L) = 2$.

If L is non-compact, then L is topologically isomorphic to $K \times \mathbb{R}^n$ where $n \geq 1$ and K is a compact connected Hausdorff abelian group of weight $w(K) \leq c$.

By Theorem 3.1, $G \times L$ is a quotient group of $\mathcal{C} \times K_1 \times L$, for some compact connected Hausdorff abelian group K_1 with weight $w(K_1) \leq c$. So $G \times L$ is a quotient group of $\mathcal{C} \times K_1 \times K \times \mathbb{R}^n$. But by Theorem 3.2, $\sigma(\mathcal{C} \times K_1 \times K \times \mathbb{R}^n) = n+1$ and so $\sigma(G \times L) \leq n+1$.

As L is a quotient of $G \times L$, Lemma 2.2 implies that $\sigma(G \times L) \geq \sigma(L)$. But from Proposition 2.4, $\sigma(L) = \sigma(\mathbb{R}^n \times K) = n+1$, and therefore, $\sigma(G \times L) \geq n+1$.

Hence $\sigma(G \times L) = n+1$. □

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