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Free Compact Groups V: Remarks on Projectivity

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Abstract. In a category with a grounding functor into a base category (such as sets, pointed spaces pointed compact Hausdorff spaces) free objects are projective—provided projectivity is defined relative to a class of epics which have right inverses in the base category. This problem arises for the category of compact groups over the category of pointed compact spaces. The epics in question are continuous group homomorphisms which have a continuous base point preserving cross section. Any deeper understanding of the relation between freeness and projectivity is, therefore, based on an understanding of these morphisms, and their study is the main objective of this article. We show among other things that a morphism of compact groups onto a compact connected abelian group C which has a continuous cross section has a homomorphic cross section and that this is definitely false for many simple not simply connected compact Lie groups C. Hence the former are homomorphic retracts of free compact groups while the latter definitely are not.

0. Freeness versus projectivity for compact groups

The category of compact groups provides an excellent example for the interplay between category theoretical concepts and procedures on the one hand and methods which, on the other hand, are characteristic for various concrete mathematical domains such as Lie group theory, representation theory, harmonic analysis, and algebraic topology. In a series of papers [4, 7, 8, 9, 11] we have illustrated what this means by discussing free compact groups. Their definition is given in purely category theoretical terms through the left adjoint of the grounding functor from the category of compact groups into the category of pointed topological spaces, and their existence can be ascertained by invoking the Left Adjoint Existence Theorem. While free objects in the catgeory of groups, i. e., the classical free groups, are somewhat bland as regards their structure theory, free compact groups, notably those over connected spaces, turn out to present rather subtle structural features. The free compact *abelian* groups form an important ingredient [8], and due to PONTRYAGIN duality, they are completely understood [7]. In the discrete case, free objects and projective objects are the same. We have pointed out [7] that this fails in the compact case where, once again, the situation is best understood on the level of the abelian compact groups.

Homological algebra, as one example, has emphasized the significance of projectivity. Therefore we want to understand projectivity in its relationship to freeness as thoroughly as possible for the category of compact groups, and the following discussion is a contribution to

This work is an original contribution and will not appear elsewhere.

this topic. [1]

Let us begin with a closer look at the relationship of freeness and projectivity in the categories most familiar, e. g. that of [abelian] groups. If FX is a free [abelian] group generated by set $X \subseteq FX$ and $f: A \to B$ is a surjective morphism of [abelian] groups. If $\mu: FX \to B$ is a morphism, then by the surjectivity of f, the set $f^{-1}(\mu(x))$ is nonempty for all $x \in X$ and thus we select an element $\alpha(x)$ in it. By the freenes of FX, the function $\alpha: X \to A$ extends uniquely to a morphism $\nu: FX \to A$. The morphisms $f \circ \nu$ and μ agree on all $x \in X$, hence on all of FX since X generates FX. Thus FX is shown to be *projective.* Clearly, in this construction we have used the Axiom of Choice. What if, for some reason—self-imposed or otherwise—we can't? One quick remedy would be that we focus on those morphisms $f: A \to B$ only, for which we know that we can select, for each element $b \in B$ and element $\sigma(b) \in f^{-1}(b)$. Thus projectivity should perhaps be formulated only relative to a certain class of epimorphisms f, and this is what turned out to be useful in homological algebra anyhow. This is the time for a precise definition and a somewhat more formal investigation of the procedure we described.

Definition 0.1. Let $\mathcal{E}$ denote a class of epics in a category $\mathcal{C}$. An object P in $\mathcal{C}$ is called an *$\mathcal{E}$-projective* and simply a *projective* if $\mathcal{E}$ is the class of all epics if for each $f: A \to B$ from $\mathcal{E}$ and each morphism $\mu: P \to B$ there is a $\nu: P \to A$ such that $\mu = f\nu$.

Now we analyze the phenomenon of the projectivity of free groups carefully in category theoretical terms: Suppose that $U: \mathcal{C} \to \mathcal{B}$ is a faithful "grounding functor". In our concrete situation $\mathcal{C}$ would be the category of [abelian] groups and $\mathcal{B}$ the category of sets with the forgetful functor U. We say that a morphism $f: A \to B$ in $\mathcal{C}$ is *$\mathcal{B}$-split* if there is a morphism $\sigma: UB \to UA$ in $\mathcal{B}$ such that $(Uf)\sigma = \mathrm{id}_{UB}$. Trivially, the retraction Uf is epic and since U is faithful, f is epic. Let $\mathcal{E}$ denote the class of $\mathcal{B}$-split morphisms. If $\mathcal{C}$ is the category of [abelian] groups and $\mathcal{B}$ the category of sets, and if the axiom of choice applies, then *every* surjective morphism is $\mathcal{B}$-split. Now the general background for our initial discussion is the following:

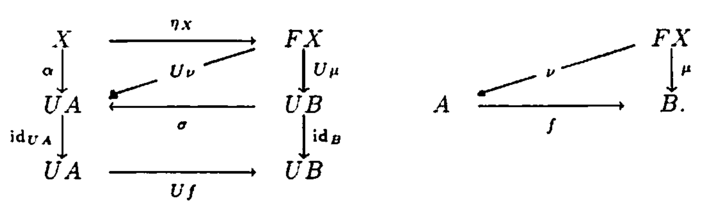
Proposition 0.2. *Suppose that $F: \mathcal{B} \to \mathcal{C}$ is left adjoint. Then FX is $\mathcal{E}$-projective for the class $\mathcal{E}$ of $\mathcal{B}$-split morphisms.*

Proof. Let $\eta_X: X \to UFX$ denote the front adjunction and set $\alpha = \sigma(U\mu)\eta_X: X \to UA$. The universal property of adjoints yields a unique morphism $\nu: FX \to A$ such that $\alpha = (U\nu)\eta_X$. Now we have

$$(U\mu)\eta_X = (Uf)\sigma(U\mu)\eta_X = (Uf)\alpha = (Uf)(U\nu)\eta_X = U(f\nu)\eta_X.$$

The uniqueness in the universal property of the adjoint now yields $\mu = f\nu$. This proves the Proposition. ∎

The following diagrams may be helpful:



[1] According to the objectives of these proceedings we want to illustrate the working of category theoretical concepts in concrete categories, at least in part, on a level accessible to a general reader, perhaps an educated graduate student.

We note that *retracts of $\mathcal{E}$-projectives are $\mathcal{E}$-projective.* Indeed if we have morphisms $\pi: P \to Q$ and $\rho: Q \to P$ with $\pi\rho = \mathrm{id}_Q$, and if P is $\mathcal{E}$-projective, then Q is also an $\mathcal{E}$-projective; for if $\mu': Q \to B$ is given, we set $\mu = \mu'\pi$, apply Proposition 0.2 to get a $\nu: FX \to A$ with $f\nu = \mu = \mu'\pi$ and obtain $\nu' = \nu\rho: Q \to B$ such that $f\nu' = \mu'\pi\rho = \mu'$. It then follows from Proposition 0.2 that any retract of a free $\mathcal{C}$-object FX is an $\mathcal{E}$-projective. On the other hand, by a fundamental property of adjoints, for every object C of $\mathcal{C}$ we have the back adjunction $\varepsilon_C: FUC \to C$ which is $\mathcal{E}$-split since $(U\varepsilon_C)\eta_{UC} = \mathrm{id}_{UC}$. Hence if C is an $\mathcal{E}$-projective, then there is a $\nu: C \to FUC$ such that $\varepsilon_C\nu = \mathrm{id}_C$. Thus every $\mathcal{E}$-projective is a retract of a free object. We summarize

Corollary 0.3. *Let $F: \mathcal{B} \to \mathcal{C}$ denote a left adjoint of a faithful grounding functor $U: \mathcal{C} \to \mathcal{B}$. An object P of the category $\mathcal{C}$ is an $\mathcal{E}$-projective for the class of $\mathcal{B}$-split epics if and only if it is a retract of a free object FX.* ∎

We are interested in the following application: Let $\mathcal{C} = \mathrm{KG}$ denote the category of compact groups and $\mathcal{B}$ the category of pointed spaces and base point preserving maps. Then the $\mathcal{B}$-split morphisms $f: A \to B$ are those morphisms of compact groups which *have a continuous cross section.*

We introduce precise terminology:

Definition 0.4. A morphism of compact groups $f: A \to B$ is said to be *topologically split* or is said *to split topologically* if there is a continuous base point preserving function $\sigma: B \to A$ with $\sigma f = \mathrm{id}_A$. Also we shall say that σ is *a continuous cross section for f*. A morphism $f: A \to B$ is called *split* or is said *to split* if there is a morphism $s: B \to A$ of compact groups such that $sf = \mathrm{id}_A$. ∎

A morphism $f: A \to B$ is topologically split, if and only if there is a compact subspace X of A such that the map $(n, h) \mapsto nh: N \times X \to A$, $N = \ker f$, is a homeomorphism. Then X is homeomorphic to B under $f|X: X \to B$ and A is topologically a product of N and B. Likewise, $f: A \to B$ is split if and only there is a compact subgroup H in A such that A is the semidirect product NH; we shall review this situation in greater detail in Section 2 below.

Topologically split morphisms are not as rare as one might think at first. MOSTERT's Cross Section Theorem (see e. g. [12], Appendix II, 1.12, pp. 317) shows that *every morphism $f: A \to B$ of compact groups is topologically split if B is zero-dimensional.* But there are even abelian compact groups such that the quotient morphism $G \to G/G_0$ modulo the identity component is not split (while it is topologically split by the preceding remarks) (see e. g. [5]).

After these remarks, for compact groups, we have to consider two kinds of projectives: (i) The $\mathcal{E}$-projectives for the class of topologically split morphisms, and (ii) the projectives, period.

We have a pretty good idea of the latter if they are connected. In fact we shall review the theory of connected projectives in Section 1 and amplify the known result that every compact connected group has a projective cover whose structure we know. This projective cover is extremely helpful in the general structure theory of compact connected groups and serves as a substitute in the case of infinite dimensional compact groups for the Lie algebra and the exponential function. Our understanding of the $\mathcal{E}$-projectives, however, depends in large measure on our understanding of topologically split morphisms. Thus category theoretical considerations lead to an internal problem on the structure of compact groups and their morphisms, namely, the study of topologically split and split morphisms. Section 2, therefore, is devoted to the splitting of morphisms, and Section 3 to the study of topologically split morphisms. We have noted, that *all* morphisms onto zero dimensional groups are topologically split. For *connected compact groups* there is a certain tendency for topologically split morphisms to be split. This is correct (although nontrivial) for compact abelian groups. As a consequence, we shall show that every compact connected abelian group is an $\mathcal{E}$-projective in the category of compact groups. This is contrast with the rather special structure of the abelian connected projectives which are exactly the duals of rational vector groups. In the context of free compact groups $\mathcal{E}$-projectivity of all compact connected abelian groups implies that every such group is a homomorphic retract of some free compact group according to Corollary 0.3. For semisimple Lie groups there are topologically split morphisms which are not split. The examples discussed in Section 3 are rather instructive. Perhaps they tell the story even better than the theorems. The obstruction seems to be in the first homotopy, but this is not sufficient to explain everything. In particular we shall show that for a simple connected compact Lie group G to be $\mathcal{E}$-projective it is necessary that the fundamental group $\pi_1(G)$ be trivial or has exponent 2 or 3. In particular, this means that a simple compact connected Lie group which is not simply connected and whose fundamental group does not have exponent 2 or 3 cannot be a homomorphic retract of a free compact group. We shall give a precise characterization of topologically split morphisms of isotypical semisimple compact connected groups; these are the building blocks from which arbitrary compact semisimple groups are constructed.

For all of this we need a recall of some general facts on the structure of compact groups. This material will be discussed in Section 1.

1. The projective cover of a compact connected group

Let us begin by recording some background material concerning compact connected groups.

Background on the structure of compact connected groups

For easy reference we record the following facts (see [10]).

A Lie algebra is said to be *compact* if it is the Lie algebra of a compact Lie group. If it is semisimple this is tantatmount to saying that its Cartan-Killing form is negative definite.

An ideal of an ideal in a Lie algebra need not be an ideal, but this is the case for compact Lie algebras. As a consequence of this fact and the fact that every compact group is a projective limit of Lie groups, if G is a connected compact group, N a connected closed normal subgroup of G and M a connected closed normal subgroup of N, then M is normal in G. The class of all compact connected groups together with morphisms with normal image form a category KGN within the full subcategory KCG of compact connected groups in the category KG of all compact groups and continuous group morphisms.

An epic in the category KG of all compact groups is surjective by a theorem of Poguntke [14]. Any epic in KCG is trivially in KGN.

If $f: G \to H$ is a surjective morphism of compact groups and H is connected, then $f(G_0) = H$ where G_0 as usual denotes the identity component of 1 in G. (Indeed $H/f(G_0)$, as a homomorphic image of G/G_0, is totally disconnected.)

In the class of all simple compact Lie algebras (i. e., simple Lie algebras which occur as the Lie algebras of compact Lie groups) we consider a set $\mathcal{L}$ of simple compact Lie algebras which contains for each simple compact Lie algebra exactly one isomorphic copy. Then $\mathcal{L}$ is a countable set, well determined up to an isomorphism of each member.

We say that G is *isotypical* (of type $\mathfrak{s}$) if every simple homomorphic image of G has $\mathfrak{s}$ as Lie algebra. It is a fact that any compact connected simple normal subgroup of an isotypical compact group of type $\mathfrak{s}$ has $\mathfrak{s}$ as its Lie algebra. More precisely, the following result is a part of the structure theory of compact connected groups (see for instance [1], [2], [5]).

Proposition 1.1. (i) *For any compact connected group G, for each $\mathfrak{s} \in \mathcal{L}$ there is a fully characteristic largest semisimple isotypical subgroup $S_\mathfrak{s}G$ of type $\mathfrak{s}$, and the commutator subgroup G' is the quotient of $\prod_{\mathfrak{s}\in\mathcal{L}} S_\mathfrak{s}G$ modulo a zero-dimensional subgroup of the center $\prod_{\mathfrak{s}\in\mathcal{L}} ZS_\mathfrak{s}G$.*

(ii) *The assignment $G \mapsto S_\mathfrak{s}G$ is the object part of a functor of the category* KGN *of compact connected groups and morphisms with normal image into the full subcategory category of compact connected semisimple isotypical groups of type $\mathfrak{s}$.*

(iii) *The assignment $G \mapsto Z_0G$, where Z_0G is the identity component of the center of G is the object part of a functor from* KGN *to the category of compact connected abelian groups.*

(iv) *The assignment $G \mapsto G'$ is the object part of a functor from the category of all compact connected groups and continuous morphisms into the category of all compact semsimple groups and continuous morphisms.*

∎

We shall call $S_\mathfrak{s}G$ *the* $\mathfrak{s}$*-component* of G. For a morphism $f: A \to B$ in KGN we call $S_\mathfrak{s}f: A_\mathfrak{s} \to B_\mathfrak{s}$ the morphism obtained by restriction and corestriction the $\mathfrak{s}$-component of f.

Let $C_\mathfrak{s}X$ denote the closed subgroup generated by all $S_{\mathfrak{s}'}G$ with $\mathfrak{s}' \neq \mathfrak{s}$. The simply connected, respectively centerfree (adjoint) compact Lie group with Lie algebra $\mathfrak{s}$ will be denoted $L_\mathfrak{s}$, respectively, $K_\mathfrak{s}$.

We notice:

Proposition 1.2. (i) *For any compact connected group G and any $\mathfrak{s} \in \mathcal{L}$ we have $G' = C_\mathfrak{s}G{\cdot}S_\mathfrak{s}G$ and $G = Z_0G{\cdot}C_\mathfrak{s}G{\cdot}S_\mathfrak{s}G$.*

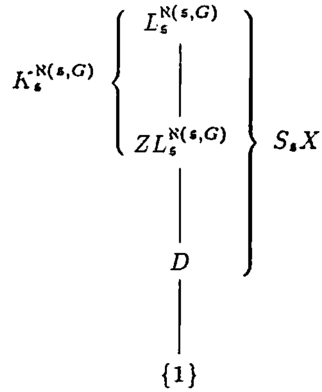
(ii) The Sandwich Theorem. *There is a unique cardinal $\aleph(\mathfrak{s}, G)$ and there are surjective morphisms $L_\mathfrak{s}^{\aleph(\mathfrak{s},G)} \to S_\mathfrak{s}G \to K_\mathfrak{s}^{\aleph(\mathfrak{s},G)}$ whose kernels are totally disconnected central subgroups, the first one of $Z(L_\mathfrak{s})^{\aleph(\mathfrak{s},G)}$ the second one of $ZS_\mathfrak{s}G$.*

(iii) *For each $\mathfrak{s} \in \mathcal{L}$ one has an isomorphism*

$$gZS_\mathfrak{s}G \mapsto gZG{\cdot}C_\mathfrak{s}G : S_\mathfrak{s}G/ZS_\mathfrak{s}G \to G/(ZG{\cdot}C_\mathfrak{s}G). \quad (1)$$

∎

The situation is best illustrated in the diagram



The Sandwich Theorem tells us in essence the structure of the $\mathfrak{s}$-component $S_\mathfrak{s}G$ of G; it depends only on the cardinal $\aleph(\mathfrak{s}, G)$ and a closed subgroup D of $Z(L_\mathfrak{s})^{\aleph(\mathfrak{s},G)}$.

On projective compact groups

We have described the projectives in KGN. The following observation shows that they remain projective in KG.

Proposition 1.3. (i) *Every projective P in* KGN *is a projective in the category* KG *of all compact groups.*

(ii) *Every group $A\times\prod_{j\in J} S_j$ with a compact abelian group A with $\widehat{A}$ a rational vector space and with simply connected compact Lie groups S_j is a projective in* KG.

Proof. For a proof of (i) let $e: A \to B$ denote a surjective morphism of compact groups and $f: P \to B$ a morphism from a projective in KGN into B. Then $B' \stackrel{\rm def}{=} \operatorname{im} f$ is a compact connected subgroup of B since P is connected. Let $A' = e^{-1}B'$ and let $e' = e|A': A' \to B'$ denote the restriction. Then e' is surjective. Since B' is connected, $e'(A'_0) = B'$. Now $e_0 = e|A'_0: A'_0 \to B'$ is a surjective morphism of compact connected groups and the corestriction $f': P \to B'$ is surjective, hence is a member of KGN. Since P is a projective in KGN there is a $F': P \to A'_0$ with $f' = e_0F'$. Then the coextension $F: P \to A$ of F' satisfies $f = eF$. This shows that P is projective as asserted.

(ii) In [2], Volume II, pp. 81ff. it is shown that the groups listed are projectives in KGN. Then they are projective in KG by (i). ∎

We focus on one functorial property in KGN, namely, the existence of a projective cover. (See [2] and [9].) A portion of the following result was formulated in Proposition 24 in [11]:

Proposition 1.4. *There is a functor P:* KGN $\to$ KGN *and a natural morphism $\tau_G: PG \to G$ such that the following properties are satisfied, where $\Delta G = \ker \tau_G$:*

(i) *Given any surjective morphism* $e: A \to B$ *of compact groups with connected* B *and any morphism* $f: PG \to B$, *then there is a morphism* $g: PG \to A$ *with* $eg = f$. *In other words,* PG *is projective in* KCG.

(ii) ΔG *is a compact zero dimensional abelian group. If* $f: X \to PG$ *is any morphism of compact connected groups such that* $\tau_G \circ f$ *is surjective, then* f *is surjective. In other words,* τ_G *is co-essential. The assignment* $G \mapsto \Delta G$ *is the object portion of a functor from the category* KGN *to the category of compact zero-dimensional abelian groups.*

(iii) *The explicit form of* PG *is given by*

$$PG = PZ_0G \times \prod_{s \in \mathcal{L}} L_s^{\aleph(s,G)}. \tag{2}$$

$$PZ_0G = (\mathbb{Q} \otimes \widehat{Z_0G})\widehat{\ }. \tag{3}$$

(iv) *If* N *is a connected compact normal subgroup of a compact connected projective group* G, *then there is a connected compact normal subgroup* H *of* G *such that* G *is the direct product of* N *and* H. *In particular,* N *and* $G/N \cong H$ *are projective.*

Proof. We start with (iii) and prove the existence of the functorially determined exact sequence

$$0 \to \Delta(G) \xrightarrow{\text{inc}} PG \xrightarrow{\tau_G} G \to 0. \tag{4}$$

By Proposition 1.1(iii, iv), the assignment $G \mapsto Z_0G \times G'$ is the object portion of a functor KGN $\to$ KGN. By 1.3(ii), the group PZ_0G is projective in KGN and $G \mapsto PZ_0G$ is a functor. The definition of the $\mathfrak{s}$-component S_sG of a compact connected group G is functiorial on KGN by Proposition 1.1(ii). By the Sandwich Theorem 1.2(ii) we have a morphism

$$\tau_{s,G}: PS_sG \stackrel{\text{def}}{=} L_s^{\aleph(s,G)} \to S_sG$$

with totally disconnected kernel. From 1.3(ii) again, PS_sG is projective in KGN. If $f: S_sG_1 \to S_sG_2$ is a morphism, then the morphism $f \circ \tau_s$ lifts to a morphism $Pf: PS_sG_1 \to PS_sG_2$; and since τ_{s,G_2} is a monic in KGN, this lifting is unique. It follows quickly that $G \mapsto S_sG \mapsto PS_sG$ is a functor. Therefore, finally, the assignment

$$G \mapsto PG \stackrel{\text{def}}{=} PZ_0 \times \prod_{s \in \mathcal{L}} PS_sG$$

is functorial on KGN. Morover, we know that there is a compact zero dimensional group $D \cong Z_0G \cap G'$ such that $G/D \cong (Z_0G)D/D \times (G'D)/D$. Also, $G'/Z(G') \cong \prod_{s \in \mathcal{L}} K_s^{\aleph(s,G)}$. Hence if we let C denote the compact zero-dimensional abelian normal subgroup $DZ(G')$, then

$$G/C \cong (Z_0G)C/C \times \prod_{s \in \mathcal{L}} K_s^{\aleph(s,G)}.$$

The natural morphism

$$f: PZ_0G \times \prod_{s \in \mathcal{L}} L_s^{\aleph(s,G)} \to (Z_0G)C/C \times \prod_{s \in \mathcal{L}} K_s^{\aleph(s,G)}$$

lifts across the quotient morphism

$$G \to (Z_0G)C/C \times \prod_{s\in\mathcal{L}} K_s^{\aleph(s,G)}$$

to a unique natural quotient morphism

$$\tau_G: PG \to G$$

with zero-dimensional compact central and functorially determined kernel $\Delta(G)$. In particular, τ_G is epic-monic in KGN. Thus we have a functorially determined exact sequence (4) with a projective PG.

In order to finish (ii) let $f: X \to PG$ denote any morphism of compact connected groups such that $\tau_g \circ f$ is surjective. Then $(\operatorname{im} f)\Delta(G) = PG$. Now $K \stackrel{\text{def}}{=} (\operatorname{im} f) \cap \Delta(G)$ is a central subgroup of PG, and the connected group PG/K is the semidirect product of the normal totally disconnected group $\Delta(G)/K$ and the connected group $(\operatorname{im} f)/K$. Hence $\Delta(G)/K = \{1\}$. Thus $\Delta(G) \subseteq \operatorname{im} f$, whence $\operatorname{im} f = PG$.

(iv) From (iii) we know that

$$G = Z_0G \times G' \quad \text{where } G' = \prod_{s\in\mathcal{L}} L_s^{\aleph(s,G)}$$

and where Z_0G is the character group of a rational vector group. Now N is a connected compact normal subgroup of G so $\operatorname{inc}: N \to G$ is a KGN-morphism, and thus $Z_0N \subseteq Z_0G$ and $S_sN \subseteq S_sG$ by 1.1(ii, iv). Thus it suffices to prove the assertion for the abelian and the semisimple isotypic case. But if G is abelian, then the surjection $f: G \to G_2$ splits since G is projective, and this suffices in the abelian case. If $G = L_s^X$ then $PG_2 = (Pf)(G)$ (by (ii) above) is isomorphic to L_s^Y; we may assume $PG_2 = L_s^Y$. For each $y \in Y$ and $x \in X$ the map $f_{yx} \stackrel{\text{def}}{=} \operatorname{pr}_y(Pf)\operatorname{copr}_x: L_s \to L_s$ is either the constant morphism 0 or and automorphism. Let $U = \{x \in X \mid (\forall y) f_{yx} = 0\}$ and $V = \{x \in X \mid (\exists y) f_{yx} \neq 0\}$. Then $X = U\dot{\cup}V$ (disjoint union!) and we identify L_s^U and L_s^V in the obvious fashion with subgroups of L_s^X. Now $L_s^U \subseteq \ker Pf$ and $(Pf)|L_s^V$ is injective. It follows that $L_s^U = \ker(Pf)$ and $H = L_s^V$ is a normal complement. Now $N = \ker f = \ker \tau_{G_2}(Pf) = (Pf)^{-1}\Delta_{G_2}$. Since N is connected and Δ_{G_2} is totally disconnected by (ii) above we have $(Pf)(N) = \{1\}$, and since Pf is surjective by (ii) we conclude $\Delta_{G_2} = \{1\}$, i. e., τ_{G_2} is an isomorphism and G_2 is projective. Thus we may write $PG_2 = G_2$ and $(Pf) = f$ whence $N = L_s^U$. The assertion is proved in the isotypical case and thus for the general situation by the preceding arguments.

∎

Corollary 1.5. *The assignment $G \mapsto PG$ is the object part of a self-functor* KCG $\to$ KCG *of the category of compact connected groups and $\tau_G: PG \to G$ is a natural transformation of P:* KCG $\to$ KCG *to the identity functor of* KCG.

Proof. Let $f: G \to H$ be a morphism in KCG. Now PG is projective in KG by Proposition 1.3(i) and $\tau_H: PH \to H$ is surjective. Hence there is a morphism $Pf: PG \to PH$ such that $\tau_H(Pf) = f\tau_G$. But τ_H is monic in the category KCG of compact connected groups because its kernel is totally disconnected by Proposition 1.4(ii). Hence Pf is uniquely determined by this equation. ∎

Notice that we do not assert that $G \mapsto \Delta G$ is functorial on KCG.

Definition 1.6. For any compact connected group G we shall call $\tau_g: PG \to G$ the projective cover of G.

2. Splitting almost split groups

Suppose that G is a topological group with a subgroup H and a normal subgroup N. The inner automorphisms by elements of H induce automorphisms of N and, as a consequence, the semidirect product $N \rtimes H$ of N by H is well defined with multiplication $(m,h)(n,k) = (m(hnh^{-1}), mn)$. Let the functions $\delta: N \cap H \to N \rtimes H$ and $\mu: N \rtimes H \to G$ be defined by $\delta(d) = (d^{-1}, d)$ and $\mu(n, h) = nh$. Then δ and μ are morphisms of topological groups and the sequence

$$(*) \qquad 0 \to N \cap H \xrightarrow{\delta} N \rtimes H \xrightarrow{\mu} G \to 0$$

is exact except possibly at G where it is exact if and only if $NH = G$.

Definition 2.1. A topological group G is said to be *almost split* over a normal subgroup N if there is a subgroup H such that the sequence $(*)$ is exact, $N \cap H$ is totally disconnected and μ is an open morphism. The group H then is called *a near complement* of N. A morphism $f: G_1 \to G_2$ of compact groups is said to be *almost [directly] split* if $\ker f$ has a [normal] near complement in G_2. ∎

The theory of connected compact groups is full of almost split situations as we know from Section 1. We shall make this precise:

Proposition 2.2. *Let $f: G \to G_2$ be a surjective morphism of compact groups onto a compact connected group with kernel N. Then G contains a near complement H which is normal if G is connected. In particular, every epic in the category* KCG *of compact connected groups is almost directly split.*

Proof. Suppose first that we have shown that the identity component G_0 contains a near complement H for $N \cap G_0$. Since G_2 is connected we have $f(G_0) = G_2$. Hence for each $g \in G$ there is a $g_0 \in G_0$ with $f(g) = f(g_0)$ whence $gg_0^{-1} \in N$ and as $g_0 \in NH$ we have $g \in NH$, i. e., $G = N \cap H$. But $N \cap H = (N \cap H) \cap H$ is totally disconnected since H is a near complement in G_0 for $N \cap G_0$. Hence H is a near complement in G for N.

It suffices now to assume that G is connected and to show that N has a normal near complement H. The map $Pf: PG \to PG_2$ is surjective by 1.4(ii). Hence the identity component PN of its kernel is projective and has a direct complement PH by 1.4(iii). Also $\tau_G(\ker(Pf) = \ker f = N$, whence $\tau_G(PN) = N$. We set $H = \tau_G(PH)$. Then $G = \tau_G(PG) = NH$ and $\tau_G^{-1}(N \cap H) = \tau_G^{-1}(N) \cap \tau_G^{-1}(H) = (PN)\Delta_G \cap (PH)\Delta_G$. If $\Delta_G(N)$ is the projection of Δ_G into the direct factor PN and $\Delta_G(H)$ is defined accordingly, then both of these groups are totally disconnected central and $(PN)\Delta \subseteq (PN)\Delta_G(H)$ and $(PH)(\Delta_G \subseteq \Delta_G(N)(PH)$ and both of the products on the right sides are direct. It follows that $\tau_G^{-1}(N \cap H) \subseteq (PN)\Delta_G(H) \cap \Delta_G(N)(PH) = \Delta_G(N)\Delta_G(H)$, and this last group is totally disconnected. As a homomorphic image of a totally disconnected compact group, $N \cap H$ is totally disconnected. This proves the assertion. ∎

Therefore, we shall now be concerned with the following question: When can a near complement be replaced by a complement? Indeed, it is sometimes possible to find a subgroup A in G such that G actually is isomorphic to the semidirect product of N and A. This question was investigated in [3]. For a formulation of the relevant result we recall that a function $c: H \to N$ between topological groups for which there is an automorphic action $(h,n) \mapsto h{\cdot}n: H \times N \to N$ of H on N is said to be *a cocycle* if it is continuous and satisfies

$$c(hk) = f(k)(k{\cdot}f(h)) \quad \text{for all} \quad h,\, k \in H.$$

Proposition 2.3. *Suppose that the sequence* $(*)$ *is exact and* μ *is open. Then the following conditions are equivalent:*

(1) *There is a subgroup* A *of* G *such that* $(n.a) \mapsto na: N \rtimes A \to G$ *is an isomorphism of topological groups.*
(2) *There is a subgroup* D *of* $N \rtimes H$ *containing* $\operatorname{im}\delta$ *such that* $(N \times \{1\}) \cap D = \{(1,1)\}$.
(3) *There is a cocycle* $c: H \to N$ *with respect to the action under inner automorphisms extending the identity function* $H \cap N \to H \cap N$.

If also H *is normal and connected, if* $H \cap N$ *is totally disconnected, and conditions* (1,2,3) *are satisfied, then* $h \mapsto c(h)^{-1}: H \to N$ *is a morphism extending the map* $d \mapsto d^{-1}: N \cap H \to N \cap H$.

Proof. For the details see [3]. The relation between D and A is given by $A = \mu(D)$ and $D = \mu^{-1}(A)$. The relation between D and c is given by $D = \operatorname{graph} c$, and $A = \{c(h)^{-1}h \mid h \in H\}$. ∎

One of the principal application of this Proposition is the following result, in essence contained in [3]:

Corollary 2.4. *Suppose that the compact group* G *has a compact connected semisimple normal subgroup* N *and an abelian central subgroup* H *such that* $G = NH$. *Then there is a compact abelian subgroup* $A \cong G/N$ *in* G *such that* $(n,a) \mapsto na: N \rtimes A \to G$ *is an isomorphism of compact topological groups.*

In particular, if G *is a compact connected group, then* G *is the semidirect product of* G' *and an abelian group* $A \cong G/G'$.

Proof. Any maximal torus T of N is a product of circles and contains $N \cap H$. Since products of circles are injective in the category of compact abelian groups, the inclusion $N \cap H \to T$ extends to a morphism $c: Z \to T \subseteq N$ which is needed for the splitting according to Proposition 2.3. This applies, in particular to any compact connected group G with its commutator subgroup $N = G'$ giving a complement $A \cong G/G'$, where $H = Z_0(G)$ is the identity component of the center. ∎

We shall now investigate circumstances under which semidirect splittings are respected by morphisms.

Proposition 2.5. *If* G_1 *is a connected normal subgroup of* G_2 *and* $G_1'A_1$ *is a semidirect splitting, then there is a closed subgroup* A_2 *in* G_2 *containing* A_1 *such that* $G_2'A_2$ *is a semidirect splitting of* G_2.

Proof. There is a morphism $c_1: Z_0(G_1) \to G_1'$ extending the identity on $G_1' \cap Z_0(G_1)$ such that $A_1 = \{c_1(z)^{-1}z \mid z \in Z_0(G_1)\}$. Let $D = G_2' \cap Z_0(G_2)$ and let $dz = d'z' \in DZ_0(G_1)$ with $d, d' \in D$, $z, z' \in Z_0(G_1)$. We notice that $G_2' \cap G_1 = G_1'$: This is true for Lie groups where it is readily verified on the Lie algebra level; it follows by approximation in the general case.

Therefore, $c_1(z')c_1(z)^{-1} = c_1(z'z^{-1}) = c_1(d'^{-1}d)$. Since

$$\begin{aligned} d'^{-1}d = z'z^{-1} &\in D \cap Z_0(G_1) \\ &= G_2' \cap Z_0(G_2) \cap Z_0(G_1) = G_2' \cap G_1 \cap Z_0(G_1) \\ &= G_1' \cap Z_0(G_1), \end{aligned}$$

we have $d'^{-1}d = c(d'^{-1}d) = c_1(z')c_1(z)^{-1}$, i. e., $dc_1(z) = d'c_1(z')$. Hence c extends to a morphism $c: DZ_0(G_1) \to G_1'$ via $c(dz) = dc(z)$ and $c(d) = d$ for $d \in D$. Now let T be any maximal torus of G_2 containing $c_1(Z_0(G_1))$. Since all maximal tori of G_2 contain D, we have $\operatorname{im} c \subseteq T$. Since T as a maximal torus in a semisimple compact connected group is a product of circles in view of the Sandwich Theorem, the corestriction $c: DZ_0(G_1) \to T$ extends to a morphism $c_2: Z_0(G_2) \to T \subseteq G_2'$ which is agrees with the identity on D. Now $A_2 = \{c_2(z)^{-1}z \mid z \in Z_0(G_2)\}$ is the desired group. ∎

Proposition 2.6. *If $f: G_1 \to G_2$ is a surjective morphism of compact connected groups, then for every semidirect decomposition $G_2 = G_2'A_2$ of G_2 there is a semidirect decomposition $G_1 = G_1'A_1$ of G_1 with $f(A_1) = A_2$. (Of course, $f(G_1') = G_2'$ is automatic).*

Proof. Let $N = G_1' \cap \ker f$. If G/N decomposes semidirectly into

$$(G_1'/N)(A/N) \quad \text{with} \quad f(A) = A_2,$$

then $A = A_0N$. As a normal subgroup of the semisimple compact connected group G_1', the group N is of the form N_0Z with some compact abelian totally disconnected group Z which is central in G_1 and a semisimple compact connected group N_0.

We write $A_0 = N_0Z_0(A_0)$ and have $A = N_0ZZ_0(A_0)$ with an abelian group $ZZ_0(A_0)$ which is central in A. Then we find an abelian compact group A^* in A such that $A = N_0A^*$ is semidirect. Let A_1 be the identity component of A^*. Now $N = N_0(N \cap A^*)$ semidirectly, and $A = NA_1$ semidirectly. Now $G = G_1'A_1$ and $G_1' \cap A_1 \subseteq G_1' \cap A \cap A_1 = N \cap A_1 = \{1\}$. Hence G_1 is decomposed semidirectly in the form $G_1'A_1$ and $f(A_1) = f(NA_1) = f(A) = A_2$. Thus we may assume from here on that $G_1 \cap \ker f = \{1\}$ i. e., that $f|G_1': G_1' \to G_2'$ is an isomorphism.

Now let $A_1 = f^{-1}(A_2)$. If $g \in G_1' \cap A_1$, then $f(g) \in f(G_1') \cap f(A_1) = G_2' \cap A_2 = \{1\}$. Hence $g \in G_1 \cap \ker f = \{1\}$, and thus $G = G_1'A_1$ is a semidirect product and $f(A_1) = A_2$. ∎

3. Topologically split epics

Our preceding results allow us to draw the following conclusion on topologically split morphisms of compact connected groups.

Proposition 3.1. *For a morphism $f: G_1 \to G_2$ of compact connected groups the following conditions are equivalent:*

(1) *f splits topologically.*

(2) *The semisimple part*

$$f': G_1' \to G_2', \quad f'(g) = g$$

and the abelian part

$$F: G_1/G_1' \to G_2/G_2', \quad F(gG_1') = f(g)G_2'$$

both split topologically.

Proof. In view of Proposition 2.6, clearly (2) implies (1). We prove that (1) implies (2). We have semidirect decompositions $G_j = G'_j \rtimes A_j$, $j = 1, 2$, with $f(A_1) = A_2$ by Proposition 2.6. Let $\sigma: G_2 \to G_1$ denote a topological cross-section for f. Define $\sigma': G'_2 \to G'_1$ and $\alpha: A_2 \to A_1$ by $\sigma'(g) = \mathrm{pr}_{G'_1}\sigma(g)$ for $g \in G_2$ and $\alpha(a) = \mathrm{pr}_{A_1}\sigma(a)$ for $a \in A_2$. Then $f\sigma'(g) = f\,\mathrm{pr}_{G'_1}\sigma(g) = \mathrm{pr}_{G'_2} f\sigma(a) = \mathrm{pr}_{G'_2}(g) = g$ and $f\alpha(a) = f\,\mathrm{pr}_{A_1}\sigma(a) = \mathrm{pr}_{A_2} f\sigma(a) = \mathrm{pr}_{A_2}(a) = a$. Hence σ' and α are the desired topological cross-sections. ∎

The situation unfortunately is more complicated for splitting in the group sense. Suppose that $f: G_1 \to G_2$ is a split morphism of compact groups with a homomorphic cross section $s: G_2 \to G_1$. Then $s(G'_2) \subseteq G'_1$, and the restriction $f': G'_1 \to G'_2$ and corestriction $s': G'_2 \to G'_1$ satisfy $f's' = \mathrm{id}_{G_2}$. Hence f' splits. We let $S: G_2/G'_2 \to G_1/G'_1$ be the induced morphism. Then $FS = \mathrm{id}_{G_2/G'_2}$ and thus F splits, too.

Some counterexamples

However, the converse may be false. In order to understand more clearly what happens we prove a lemma:

Lemma 3.2. *Let N denote a semisimple compact connected normal subgroup of a compact connected group G. There is a unique compact connected normal subgroup M such that $G' = NM$ and $N \cap M$ is totally disconnected and central in G. Let $Z(M)$ and $Z(N)$ the centers of M and N, respectively, and write $\Delta = Z_0 \cap G'$. Then*

(i) *the following conditions are equivalent:*
   (1) *G is a semidirect product NB, $N \cap B = \{1\}$.*
   (2) *There is a morphism $\alpha: MZ_0 \to N$ extending the inclusion map $Z(N) \cap Z(M)\Delta$.*

(ii) *If α exists as in (2) and is surjective, then*

$$M \cap N \cap Z_0 = Z(M) \cap Z(N) \cap \Delta = \{1\}.$$

Proof. The existence of M follows from Proposition 1.4(iv). We note that $N \cap (MZ_0)$ is totally disconnected in view of the structure theoy, hence is central. Hence $N \cap (MZ_0) \subseteq Z(N) \cap Z(MZ_0) = Z(N) \cap Z(M)Z_0$. If $n = mz$ with $n \in N$, $m \in M$, and $z \in Z_0$, then $z = m^{-1}n \in MN \cap Z_0 = G' \cap Z_0 = \Delta$. Hence $N \cap (MZ_0) \subseteq Z(N) \cap Z(M)\Delta$, and the reverse inclusion is trivial, so equality holds. The equivalence of (1) and (2) is now simply a consequence of Proposition 2.3. Thus (i) is proved. For a proof of (ii) note that the surjective morphism $\alpha: MZ_0 \to N$ must map Z_0 to the identity. Hence $\alpha(M \cap Z_0) = \{1\}$. On the other hand, $\alpha(m) = m$ for $m \in Z(N) \cap Z(M)\Delta$ by (2). Hence (ii) follows. ∎

Example 3.3. There is a compact connected Lie group G with the following properties:

(i) $G/G' \cong \mathsf{T} = \mathsf{R}/\mathsf{Z}$.

(ii) There is a compact connected normal subgroup N contained in G' such that N is a semidirect factor in G' but not in G.

(iii) If $f: G \to G/N$ denotes the quotient morphism, then f' splits and the induced morphism $G/G' \to (G/N)/(G/N)'$ is an isomorphism.

(iv) f splits topologically but not algebraically. In particular, G/N is not an $\mathcal{E}$-projective in KG for the class of topologically split morphisms.

(v) The smallest example G of this kind has dimension 7 with $\dim G/N = 4$.

Proof. Let L denote a simple simply connected Lie group with cyclic center Z of order n with generator z. Denote with $t \in \mathbb{T}$ an element of order n. In the group $L \times L \times \mathbb{T}$ consider the central subgroup D generated by the elements $(z, z, 0)$ and $(1, z, t)$. Set $G = (L \times L \times \mathbb{T})/D$. Then $G' = (L \times L \times \{0\})D/D \cong (L \times L)/\{(c, c) | c \in Z\}$ contains the normal subgroups $N = (L \times \{1\} \times \{0\})D/D \cong L$ and $M = (\{1\} \times L \times \{0\})D/D \cong L$, and the diagonal subgroup $C = \{(d, d, 0) | d \in L\}D/D \cong L/Z$. We claim that G' is the semidirect product of N and H. Clearly $NC = G'$. In order to show $N \cap C = \{1\}$ consider $(u, v, w)D \in N \cap C$. Then $(u, v, w) \in (L \times \{1\} \times \{0\})D \cap \{(d, d, 0) | d \in L\}D = \{(x, z^m, p{\cdot}t) | x \in L, m, p \in \mathbb{Z}\} \cap \{(d, dz^p, p{\cdot}t) | d \in L, p \in \mathbb{Z}\} = D$ which proves the claim.

Also

$$\begin{aligned} N \cap Z_0 &= (L \times \{1\} \times \{0\})D/D \cap (\{1\} \times \{1\} \times \mathbb{T})D/D \\ &= (L \times Z \times \mathbb{Z}{\cdot}t \cap Z \times Z \times \mathbb{T})/D \\ &= (Z \times Z \times \mathbb{Z}{\cdot}t)/D \cong Z. \end{aligned}$$

Thus $N \cap MZ_0 \supseteq N \cap Z_0 \neq \{1\}$ if $Z \neq \{1\}$.

Likewise

$$\begin{aligned} M \cap Z_0 &= ((\{1\} \times L \times \{0\})D/D \cap (\{1\} \times \{1\} \times \mathbb{T})D)/D \\ &= (Z \times L \times \mathbb{Z}{\cdot}t \cap \mathbb{Z} \times Z \times \mathbb{T})/D \\ &= (Z \times Z \times \mathbb{Z}{\cdot}t)/D = N \cap Z_0. \end{aligned}$$

Thus $M \cap N \cap Z_0 \cong Z \neq \{1\}$.

If we let $f: G \to G/N$ denote the quotient morphism, then $f': G' \to G'/N$ splits by the preceding. Now $G/G' \cong (G/N)/(G'/N) = (G/N)/(G/N)'$ by the first isomorphy theorem. In particular, f splits topologically since $G = G'A$ semidirectly with a suitable abelian group $A \cong G/G'$ by 2.4, and thus topologically $G = N \times C \times A$.

However, N does not admit a semidirect group complement in G. For a proof we note that by Lemma 3.2 all subgroups of G complementary to N are classified by a morphism $\alpha: MZ_0 \to N \cong L$ extending the identity map of $N \cap MZ_0 \cong Z$. Then α cannot be constant if $Z \neq \{1\}$. In this case α is necessarily surjective. Hence by Lemma 3.2(ii) we have $M \cap M \cap Z_0 = \{1\}$, a contradiction.

An $\mathcal{E}$-morphism onto an $\mathcal{E}$-projective splits. Thus G/N cannot be $\mathcal{E}$-projective.

The smallest example is given by $L = \mathrm{SU}(2)$ with $Z = \{1, -1\}$ of order 2. In this case $\dim G = 3 + 3 + 1 = 7$

■

This example shows, in particular, that there are morphisms which are topologically split but are not split. One might surmise that this may not occur with morphisms between semisimple groups. However this is not the case as the following example illustrates:

Example 3.4. There is a surjective morphism $f: G_1 \to G_2$ of semisimple isotypical compact connected Lie groups which splits topologically but not algebraically. In particular, G_2 is not an $\mathcal{E}$-projective for the class of topologically split morphisms. One example is given by $G_2 = PSU(6)$ and G_1 locally isomorphic to $SU(6)^2$.

Proof. We let L again denote a simple simply connected Lie group with nontrivial cyclic center Z. We consider $\Delta = \{(z^a, z) | z \in Z\} \subseteq L^2$ with a natural number a. We set $G_1 = L^2/\Delta$,

$G_2 = L/Z$. We let $f: G_1 \to G_2$ denote the morphism induced by the projection $p: L^2 \to L$ onto the last component. The kernel N then equals $(L \times \{1\})\Delta/\Delta = (L \times Z)/\Delta \cong Z$. The unique supplementary normal subgroup is $M = (\{1\} \times L)\Delta/\Delta = (Z^a \times L)/\Delta \cong L/Z[a]$ where $Z[a] = \{x \in Z \mid x^a = 1\}$. Moreover, $N \cap M = (Z^a \times Z)/\Delta \cong Z/Z[a]$. By Proposition 2.3, the complements for N in G_1 are characterized by morphisms $\alpha: M \to N$ extending the inclusion $N \cap M \to N$. Since $N \cap M$ is nontrivial if $Z[a] \neq Z$, any such morphism is nontrivial. But a nontrivial morphism $L/Z[a] \to L$ must an isomorphism, and $Z[a] = \{1\}$. Hence if $\{1\} \neq Z[a] \neq Z$, such an α cannot exist. Whenever the order of Z is not a prime, a number a with this property exists. An example is $L = SU(6)$.

On the other hand let us consider the continuous function $\tilde{\sigma}: L \to L^2$ given by $\tilde{\sigma}(v) = (v^a, v)$. Then $p \circ \tilde{\sigma} = \mathrm{id}_L$. Also, if $z \in Z$, then $\tilde{\sigma}(zv) = \big((zv)^a, zv\big) = (v^a, v)(z^a, z) \in \tilde{\sigma}(v)\Delta$ since z is central. Hence $\tilde{\sigma}$ induces a base point preserving continuous function $\sigma: G_2 \to G_1$ given by $\sigma(vD) = (v^a, v)\Delta$ with is a continuous cross section for f. ∎

The abelian case

However, the abelian situation is radically different:

Proposition 3.5. *A topologically split morphism $f: A \to B$ of compact connected abelian groups splits.*

Proof. Let $\sigma: B \to A$ denote the topological cross section. We denote with $H^*(X) = H^*(X, \mathbb{Z})$ the integral Čech cohomology of a compact space X. The relation $f\sigma = \mathrm{id}_B$ induces the relation

$$H^*(\sigma)H^*(f) = H^*(f\sigma) = H^*(\mathrm{id}_B) = \mathrm{id}_{H^*(B)}: H^*(B) \to H^*(A)$$

in Čech-cohomology over the integers. We specialice to dimension one and obtain a split exact sequence

$$0 \to \ker H^1(\sigma) \xrightarrow{\text{inc}} H^1(B) \underset{f^*}{\overset{\sigma^*}{\rightleftarrows}} H^1(A) \to 0$$

with the notation $\phi^* = H^1(\phi)$, designating a morphism of discrete torsion free abelian groups. The dual is a split sequence of compact connected abelian groups

$$(\dagger) \qquad 0 \to H^1(A)\hat{\ } \underset{\widehat{f^*}}{\overset{\widehat{\sigma^*}}{\rightleftarrows}} H^1(B)\hat{\ } \xrightarrow{\widehat{\text{inc}}} (\ker H^1(\sigma))\hat{\ } \to 0.$$

There is, however, a natural isomorphism $\widehat{G} \to H^1(G)$ between the character group of a compact connected group and its first integral cohomology group. (See e. g., [13].) Consquently, by Pontryagin duality, there is a natural isomorphism $H^1(A)\hat{\ } \to A$ and $H^1(B)\hat{\ } \to B$ by which $H^1(f)\hat{\ }$ becomes identified with $f: A \to B$. The split exact sequence $(\dagger)$ therefore proves the Proposition. ∎

Before we generalize this result, we need a reduction:

Lemma 3.6. (i) *If $f: A \to B$ is a surjective morphism of compact groups and A_1 is a subgroup of A with $f(A_1) = B$, then f splits [topologically] if $f|A_1: A_1 \to B$ splits [topologically].*

(ii) *If $f: A \to B$ is a morphism of compact groups onto a connected group and $f_0: A_0 \to B$ denotes the restriction, then f_0 splits topologically if f splits topologically and f splits if f_0 splits.*

Proof. (i) Let $K = \ker f$. It suffices to find a compact subgroup [subspace] H of A such that $A = KH$ and $(k,h) \mapsto kh: K \times H \to A$ is a homeomorphism, for then $f|H: H \to B$ is an isomorphism [homeomorphism] and $s = j(f|H)^{-1}: B \to A$ with the inclusion $j: H \to A$ is the required homomorphic [continuous] cross-section. Now let $H \subseteq A_1$ denote a [topological] complement for $K \cap A_1$ in A_1; then $K \cap H = (K \cap A_1) \cap H = \{1\}$ and $f(A_1) = B$ implies $KH = A$. Thus $kh = k'h'$ implies $k'^{-1}k = h'h^{-1} \in H \cap K = \{1\}$, i. e., $k = k'$ and $h = h'$. Hence H is a [topological] complement for K in A.

(ii) f splits topologically, then any base point preserving cross-section $\sigma: B \to A$ maps B into A_0, whence f_0 splits topologically. The second assertion follows from (i) above. ∎

Lemma 3.7. *If $f: G_1 \to G_2$ is a morphism with a topological base point preserving cross-section $\sigma: G_2 \to G_1$ and if H_2 is any closed subgroup of G_2, then the restriction and corestriction $f^{-1}(H_2) \to H_2$ is a topologically split morphism.*

Proof. Since σ is a base point preserving cross section of f we know that $\sigma(H_2) \subseteq f^{-1}(H_2)$. Hence σ restricts and corestricts to a base point preserving map $H_2 \to f^{-1}(H_2)$ which is a cross-section for the morphism $f^{-1}(H_2) \to H_2$ induced by f. ∎

Theorem 3.8. (i) *A topologically split morphism of compact groups $f: G_1 \to G_2$ onto an abelian connected group splits.*

(ii) *If $f: G_1 \to G_2$ is any topologically split morphism of compact groups and T is a connected abelian subgroup of G_2 then the restriction and corestriction $T_1 \to T$, $T_1 = f^{-1}(T)$ splits. Moreover, if S is any maximal connected abelian subgroup of T_1, then the restriction $S \to T$ splits.*

Proof. By Lemma 3.6(ii) it is no loss of generality to assume that G_1 is connected. Then by 2.4, the group G_1 is a semidirect product $G'A$ with an abelian group A with $f(A) = G_2$ since f is surjective and G_2 is abelian. As the induced morphism $f|A: A \to G_2$ is equivalent to the induced morphism $G_1/G_1' \to G_2/G_2'$ and thus splits topologically by Proposition 3.1, it splits by Proposition 3.5. Hence there is a compact subgroup B of A such that $A = (N \cap A)B$ is direct with $N = \ker f$. Now $NB = N(N \cap A)B = NA = G$ and $N \cap B = N \cap A \cap B = \{1\}$. Hence NB is a semidirect decomposition and the inverse of the isomorphism $f|B: B \to G_2$ produces the required homomorphic cross-section.

(ii) By Lemma 3.7 , the restriction and corestriction $T_1 \to T$ is topologically split. Hence it splits by (i). Thus $T_1 = NA$ with $N = \ker f$ and an abelian group $A \cong T$. Let S be a maximal connected abelian subgroup of T_1 containing A. Then $S = (S \cap N)A$ is a direct decomposition, and thus the induced morphism $S \to T$ splits. Since all maximal connected abelian subgroups of T_1 are conjugate, the assertion follows. ∎

Theorem 3.9. *Every compact connected abelian group is an $\mathcal{E}$-projective for the class of topologically split epics in the category* KG *of compact groups. In particular, every such group is a homomorphic retract of a free compact abelian group.*

Proof. Suppose that P is any compact connected abelian group and $f: P \to B$ a morphism of compact groups. Suppose that $e: A \to B$ is a topologically split morphism. Let $e_1: e^{-1}(f(P)) \to f(P)$ is a topologically split morphism by Lemma 3.7. Hence it splits by Theorem 3.8(i). If

$s\colon f(P) \to e^{-1}(f(P))$ is a homomorphic cross-section for e_1, then $F\colon P \to A$, $F(g) = s(f(p))$ satisfies $ef = f$. This proves that P is a $\mathcal{E}$-projective. The last assertion is a consequence of Corollary 0.3. ∎

Reducing topological splittings

In the remainder of the section we shall prove that the topological splitting of a morphism of compact connected groups $f\colon A \to B$ reduces to the topological splitting of the isotypical components $S_s f_0\colon S_s A_0 \to S_s B$. Moreover we shall precisely describe the topological and the algebraic splitting of the isotypical components.

Throughout the following discussion we let $\sigma\colon B \to A$ denote a continuous cross section for f.

Since the abelian case is settled, we now turn to the general semisimple case.

Proposition 3.10. *Let $f\colon A \to B$ be a topologically split homomorphism with a continuous cross-section $\sigma\colon B \to A$, and assume that $A = A'$ and $B = B'$ are semisimple. Then there is a continuous. base point preserving map $\phi\colon PB \to PA$ such that*

(i) $Pf\sigma = \mathrm{id}_{PB}$.

(ii) $\sigma\tau_B = \tau_A\phi$.

(iii) $\phi(tx) = \phi(t)\phi(x)$ *for all* $x \in PB$, $t \in \ker\tau_B$. *and there is a commuting diagram of exact sequences*

$$\begin{array}{ccccccccc} 0 & \to & \Delta_B & \xrightarrow{\text{inc}} & PB & \xrightarrow{\tau_B} & B & \to & 0 \\ & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\sigma} & & \\ 0 & \to & \Delta_A & \xrightarrow{\text{inc}} & PA & \xrightarrow{\tau_A} & A & \to & 0 \end{array}$$

where $\Delta_X = \ker\tau_X$ *and where* $\psi = \phi|\Delta_B$ *is the restriction and corestriction of* ϕ.

(vi) *The morphism* $\pi = Pf|\Delta_A \to \Delta_B$ *splits and* $\pi\psi = \mathrm{id}_{\Delta_B}$. *In particular,* Δ_A *is a direct product* $(\ker\pi)(\mathrm{im}\,\psi)$.

Proof. (i) Let $q_a\colon A' \to A/Z(A)$ denote the natural homomorphism $a \mapsto aZ(A)$. Then $\pi_A = q_A\tau_A\colon PA \to A/Z(A)$ is equivalent to the morphism

$$\prod_{s\in\mathcal{L}} L_s^{\aleph(s,A)} \to \prod_{s\in\mathcal{L}} K_s^{\aleph(s,A)}$$

induced by the universal covering morphisms $p_j\colon L_s \to K_s$ for $j \in J$ in the appropriate simultaneous index set for both products. We identify π_A with this morphism $\prod_{j\in J} p_j$ and consider the continuous base point preserving function $\psi = q_A\sigma\tau_B\colon PB \to A/Z(A)$ with the topological cross section $\sigma\colon B \to A$. As a product of simply connected spaces L_s, the space PB is simply connected. Hence every morphism $\psi_j = p_j\psi\colon PB \to L_s$ lifts uniquely to a base point preserving map $\phi_j\colon PB \to L_s$ satisfying $p_j\phi_j = \psi_j$. Hence there is a unique base point preserving map $\phi\colon PB \to PA$ with $\pi_A\phi = \psi$. Now we compute

$$\begin{aligned} \pi_B(Pf)\phi &= \big(f/Z(A)\big)\pi_A\phi = \big(f/Z(A)\big)\psi \\ &= \big(f/Z(A)\big)q_A\sigma\tau_B = q_B f\sigma\tau_B' \\ &= q_B\tau_B = \pi_B. \end{aligned}$$

Thus $g \mapsto g^{-1}(Pf)\phi(g): PB \to B/Z(B)$ is a base point preserving continuous function mapping the connected space PB into the totally disconnected kernel $\ker \pi_B = Z(B)$. Hence it is constant. Thus

$$(Pf)\phi = \mathrm{id}_{PB}.$$

(ii) The base point preserving continuous maps $\alpha \stackrel{\mathrm{def}}{=} \tau_A\phi$ and $\beta \stackrel{\mathrm{def}}{=} \sigma\tau_B$ satisfy $q_A\alpha = \psi = q_A\beta$ by the definition of ϕ. Hence $g \mapsto \alpha(g)^{-1}\beta(g): PB \to \ker q_A = Z(A)$ is a well defined base point preserving map from a connected space into a totally disconnected one and is therefore constant. Hence

$$\tau_A\phi = \sigma\tau_B.$$

(iii) The relation $\tau_A\phi(tx) = \sigma\tau_B(tx) = \sigma\tau_B(x) = \tau_A\phi(x)$ shows that

$$\phi(tx)\phi(x)^{-1} \in \ker \tau_A = \Delta_A.$$

For each fixed $t \in \Delta_B$ the continuous base point preserving function $x \mapsto \phi(tx)\phi(x)^{-1} : PB \to \Delta_A$ from a connected space to a totally disconnected one is necessarily constant. The base point 1 is mapped to $\phi(t1)\phi(1)^{-1} = \phi(t)$. Thus $\phi(tx)\phi(x)^{-1} = \phi(t) = \psi(t)$ for all $t \in \Delta_B$ and $x \in PB$. Hence (iii) is proved.

(vi) The relation $Pf \circ \phi = \mathrm{id}_{PB}$ implies $\pi \circ \psi = \mathrm{id}_{\Delta_B}$ by simple restriction and corestriction. Thus $\Delta_A = (\ker \pi)(\mathrm{im}\,\psi)$ is a semidirect product decomposition. But Δ_A is central in PA, and thus we have a direct product decomposition. ∎

Lemma 3.11. *For a topologically split morphism $f: A \to B$ of compact connected groups there is a continuous cross section $\phi_s: S_sPB \to S_sPA$ for the $\mathfrak{s}$-component $(Pf)_s$ and a continuous cross section $\sigma_s: S_sB \to S_sA$ for the $\mathfrak{s}$ component $f_s: S_sA \to S_sB$ such that $\sigma_s(S_s\tau_B) = (S_s\tau_A)\phi_s$.*

Proof. The morphism $Pf: PA \to PB$ respects the $\mathfrak{s}$-components and may by uniquely written in the form

$$\prod_{s\in\mathcal{L}} f_s: \prod_{s\in\mathcal{L}} L_s^{\aleph(s,A)} \to \prod_{s\in\mathcal{L}} L_s^{\aleph(s,B)}.$$

If $\mathrm{copr}_s: L_s^{\aleph(s,B)} \to PB$ is the natural embedding then

$$\phi_s \stackrel{\mathrm{def}}{=} \mathrm{pr}_s\,\phi\,\mathrm{copr}_s: L^{\aleph(s,B)} \to L^{\aleph(s,A)}$$

is a well defined base point preserving continuous map and

$$Pf_s\phi_s = \mathrm{id}: L^{\aleph(s,B)} \to L^{\aleph(s,B)}$$

as we see from the commutative diagram

$$\begin{array}{ccccc}
L^{\aleph(s,B)} & \xrightarrow{\phi_s} & L^{\aleph(s,A)} & \xrightarrow{Pf_s} & L^{\aleph(s,B)} \\
\mathrm{copr}_s \downarrow & & \uparrow \mathrm{pr}_s & & \uparrow \mathrm{pr}_s \\
\prod_{s\in\mathcal{L}} L^{\aleph(s,B)} & \xrightarrow{\phi} & \prod_{s\in\mathcal{L}} L^{\aleph(s,A)} & \xrightarrow{f} & \prod_{s\in\mathcal{L}} L^{\aleph(s,B)}.
\end{array}$$

We now abbreviate $A_s = S_s(A) = \tau_A(L_s^{\aleph(s,B)})$ and B_s accordingly so that $f_s = S_sf: A_s \to B_s$. Let $j_A: A_s \to A$ denote the inclusion. We define $\sigma_s' = \sigma j_B: B_s \to A$ and let $\tau_{B,s}: L_s^{\aleph(s,B)} \to B_s$ denote the corestriction of $\tau_B \circ \mathrm{copr}_s$. Then $\sigma'_s \tau_{B,s} = \sigma \tau_B \mathrm{copr}_s^{(B)} = \tau_A \phi \mathrm{copr}_s^{(B)} = \tau_A \mathrm{copr}_s^{(A)} \phi_s = j_A \tau_{A,s} \phi_s$, and this shows that the image of σ'_s is contained in A_s. Hence there is a well defined corestriction $\sigma_s: B_s \to A_s$ such that

$$\sigma_s \tau_{B,s} = \tau_{A,s} \phi_s \quad \text{and} \quad j_A \sigma_s = \sigma j_B,$$

where the second equation follows from $j_A \sigma_s \tau_{B,s} = \sigma'_s \tau_{B,s} = \sigma j_B \tau_{B,s}$ and the surjectivity of $\tau_{B,s}$. Now

$$\begin{aligned} j_b f_s \sigma_s \tau_{B,s} &= f j_A \sigma_s \tau_{B,s} \\ &= f \sigma j_B \tau_{B,s} = j_B \tau_{B,s}, \end{aligned}$$

and thus the surjectivity of $\tau_{B,s}$ and injectivity of j_B show

$$f_s \sigma_s = \mathrm{id}_{B_s}. \qquad \blacksquare$$

We now concentrate on topologically split morphisms of isotypical groups.

Lemma 3.12. *Let $\chi: L^Y \to L^U$ denote a morphism. Then $\chi \in \mathrm{Hom}(L^Y, L^U)$ is characterized by a partial function $\nu: V \to Y$, $V \in U$ and a function $\alpha: V \to \mathrm{Aut}\, L$, $(\alpha_u)_{u \in V} \in (\mathrm{Aut}\, L)^V$ which determine $\chi \stackrel{\mathrm{def}}{=} \chi_{\nu,\alpha}$ according to*

$$(5) \qquad \chi\big((a_y)_{y \in Y}\big) = (b_u)_{u \in U} \quad \text{with} \quad b_u = \begin{cases} \alpha_u(a_{\nu(u)}) & \text{if } u \in V, \\ 1 & \text{if } u \in U \setminus V. \end{cases}$$

Each one of these induces a morphism $Z^Y \to Z^U$ by restriction and corestriction.

Proof. For each $u \in U$ the composition $\mathrm{pr}_u \circ \chi: L^Y \to L$ is either the constant morphism 0 or else there is a unique element $\nu(u) \in Y$ and an automorphism $\alpha_u: L \to L$ such that $\alpha_u = \mathrm{pr}_u \circ \chi \circ \mathrm{copr}_{\nu(u)}$. In a diagram:

$$\begin{array}{ccc} L^Y & \xrightarrow{\quad \chi \quad} & L^U \\ {\scriptstyle \mathrm{copr}_{\nu(u)}} \big\uparrow & & \big\downarrow {\scriptstyle \mathrm{pr}_u} \\ L & \xrightarrow[\quad \alpha_u \quad]{} & L. \end{array}$$

Let us set $V = \{u \in U \mid \mathrm{pr}_u \circ \chi \neq 0\}$. Then (5) holds. $\blacksquare$

Proposition 3.13. (i) *Let $f: A \to B$ be a surjective morphism of isotypical compact connected groups. Then $PA = L^X$ and $PB = L^Y$ with a simply, simply connected Lie group L with Lie algebra $\mathfrak{s}$ and sets $X = \aleph(\mathfrak{s}, A)$ and $Y = \aleph(\mathfrak{s}, B)$ such that X is a disjoint union $U \dot\cup Y$ such that we can write $PA = L^U \times L^Y$ and have exact sequences*

$$\begin{array}{ccccccccc} 0 & \to & \Delta_A & \xrightarrow{\quad \mathrm{inc} \quad} & L^U \times L^Y & \xrightarrow{\quad \tau_A \quad} & A & \to 0 \\ 0 & \to & \Delta_B & \xrightarrow{\quad \mathrm{inc} \quad} & L^Y & \xrightarrow{\quad \tau_B \quad} & B & \to 0. \end{array}$$

There is a family $(\alpha_y)_{y \in Y} \in (\mathrm{Aut}\, L)^Y$ and a bijection $\rho: Y \to Y$ such that

$$Pf\big((g_u)_{u \in U}, (g_y)_{y \in Y}\big) = (\alpha_y g_{\rho(y)})_{y \in Y} \stackrel{\mathrm{def}}{=} f_2\big((g_u)_{y \in Y}\big)$$

and $Pf(\Delta_A) \subseteq \Delta_B$.

(ii) *f has a homomorphic cross section $s: B \to A$ if and only if there is a subset $V \subseteq U$, a function $(\alpha'_u)_{u\in V} \in (\operatorname{Aut} L)^V$ and a function $\nu: V \to Y$ such that*

$$(Ps)((g_y)_{y\in Y}) = \left(\chi_{\nu,\alpha'}((g_y)_{y\in Y}), \left(\alpha^{-1}_{\rho^{-1}(y)}(g_{\rho^{-1}(y)})\right)_{y\in Y}\right) = \left((h_u)_{u\in Y}, \left(\alpha^{-1}_{\rho^{-1}(y)}(g_{\rho^{-1}(y)})\right)_{y\in Y}\right)$$

with

$$h_u = \begin{cases} \alpha'_u(g_{\nu(u)}) & \text{if } u \in V, \\ 0 & \text{if } u \in U \setminus V, \end{cases}$$

defines a homomorphic cross-section for Pf, and that $(Ps)(\Delta_B) \subseteq \Delta_A$.

(iii) *f has a continuous cross section $\sigma: B \to A$ if and only if there is a continuous base point preserving function $\kappa: L^Y \to L^U$ such that the function $\phi: L^Y \to L^U \times L^Y$ defined by $\phi(x) = (\kappa(x), f_2^{-1}(x))$ satisfies $\phi(\Delta_B) = \Delta_A$.*

Proof. (i) Since $A = A_\mathfrak{s}$ and $B = B_\mathfrak{s}$ are isotypic the abbreviations $L = L_\mathfrak{s}$ and $\aleph(\mathfrak{s}, A) = X$, $\aleph(\mathfrak{s}, B) = Y$ yield $PA = L^X$ and $PB = L^Y$. If Z is the finite center of L, then $\delta_A \subseteq Z^X$ and $\delta_B \subseteq Z^Y$. Every surjective morphism from a projective object splits. Hence we now write $X = U\dot{\cup}Y$ and $L^X = L^U \times L^Y$ where $L^U = \ker Pf$ and thus, writing the elements of PA as pairs $(a_1, a_2) \in L^U \times L^Y$, we have $Pf(a_1, a_2) = f_2(a_2)$ with an isomorphism $f_2: L^Y \to L^Y$ which by Lemma 3.12, is necessarily of the form $(a_y)_{y\in Y} \mapsto (\alpha_y(a_{\rho(y)}))_{y\in Y}$ with a bijection ρ of Y and automorphisms α_y of L. It is clear from the naturality of τ that $Pf(\Delta_A) \subseteq \Delta_B$. Note that $f_2^{-1}((b_y)_{y\in Y}) = (\alpha^{-1}_{\rho^{-1}(y)} h_{\rho^{-1}(y)})$

(ii) If $s: B \to A$ is a homomorphic cross section for f, then Ps is a homomorphic cross section for Pf. If $(Ps)(x) = (a_1, a_2)$ then $x = (Pf)(Ps)(x) = (Pf)(a_1, a_2) = f_2(a_2)$, whence $a_2 = f_2^{-1}(x)$. Also, $a_1 = \chi(x)$ for some morphism $\chi: L^Y \to L^U$. It follows from Lemma 3.12 that $(Ps)(\Delta_B) \subseteq \Delta_A$. It is clear that any morphism $x \mapsto (\chi(x), f_2^{-1}(x))$ respecting the Δ's will be a morphism Ps for a homomorphic cross section $s: B \to A$ for f.

(iii) Suppose that f has a continuous cross-section. By Proposition 3.11 there is a continuous cross section ϕ for Pf. Then as in the proof of (ii) we conclude that $\phi(x) = (\kappa(x), f_2^{-1}(x))$ with a base point preserving continuous function $\kappa: L^Y \to L^U$.

From 3.11 we know $\phi(tx) = \phi(t)\phi(x)$ for $t \in \Delta_B$ and $x \in L^Y$, and this yields

$$(\kappa(tx), f_2^{-1}(tx)) = \phi(tx) = \phi(t)\phi(x) = (\kappa(t), f_2^{-1}(t))(\kappa(x), f_2^{-1}(x)) = (\kappa(t)\kappa(x), f_2^{-1}(tx)).$$

Thus

(6) $$\kappa(tx) = \kappa(t)\kappa(x) \quad \text{for all} \quad t \in \Delta_B \subseteq Z^Y \text{ and } x \in L^Y,$$

where

$$(\kappa(t), f_2^{-1}(t)) \in \Delta_A \quad \text{for all} \quad t \in \Delta_B.$$

Conversely, any base point preserving continuous map of the form $x \mapsto (\kappa(x), f_2^{-1}(x))$ respecting the Δ's will be the ϕ for a continuous cross section $\sigma: B \to A$ for f. This completes the proof. ∎

If f has a continuous cross section, then the group $\Delta_A \subseteq Z^U \times Z^Y$ is a direct product of the subgroups $\ker \pi = \Delta_A \cap (L^U \times \{1\})$ and $\operatorname{im} \psi = \{(\kappa(x), f_2^{-1}(x))\ x \in \Delta_B\}$.

The problem of converting a topological splitting of a morphism of isotypic groups into an algebraic splitting, after 3.13, is the following:

For a continuous function $\kappa: L^Y \to L^U$ *satisfying* (6) *find a partial function* $\nu: V \to Y$, $V \subseteq U$ *and a function* $\alpha: V \to \mathrm{Aut}(L)$ *such that* $\kappa|\Delta_B = \chi_{\nu,\alpha}|\Delta_B$.

Our Example 3.3 shows that this is not always possible. We shall now render this example more precise. Let Y and U be singleton. Let Δ_B be a central subgroup of L and set $B = L/\Delta_B$. Let a denote a natural number and define $\kappa: L \to L$ by $\kappa(x) = x^a$. Then (6) is satisfied. Set $\Delta_A = \{(\kappa(z), z) | z \in \Delta_B\}$ and $A = (L \times L)/\Delta_A$. The projection $L \times L \to L$ onto the second factor induces a topologically split morphism $f: A \to B$. In order for it to be split we must find a morphism $\chi: L \to L$ with

$$\chi(z) = \kappa(z) = z^a \quad \text{for } z \in \Delta_B. \tag{7}$$

We shall now search for groups L for which there is a natural number a such that (7) can be satisfied for suitable endomorphism χ of L. If $\Delta_B^a = \{1\}$, then the constant χ will satisfy (7). If $z^a = z$, for all z, i. e., $\Delta_B^{a-1} = \{1\}$, then $\chi = \mathrm{id}_L$ satisfies (7). We now look for those L such that for each nonconstant endomorphism $\chi \neq \mathrm{id}_L$ there is a natural number $a > 1$ such that (7) is satisfied. Then χ must be an isomorphism because of the simplicity of L. In particular, $z \mapsto z^a$ has to be a nonidentity automorphism of Δ_B. The automorphisms induced on the center of a simple simply connected Lie group are known and catalogued (see e. g. [15]). They come from automorphisms of the Dynkin diagram; if the center is nontrivial the automorphism groups induced on the center have order 2 or are (in the case D_4) isomorphic to S_3. Thus we now inspect the list whether we find outer automorphisms χ of L which on the center induce a map of the form $z \mapsto z^a$.

Type A_{n-1} represented by $L = \mathrm{SU}(n)$, $\Delta_B \cong Z(n)$, $\chi(z) = z^{-1} = z^a$ with $a \equiv -1 \pmod n$. If $n > 3$ then we find always natural numbers $a > 1$ with $a \not\equiv -1 \pmod n$.

Types B and C have no outer automorphisms.

Type D_n, $n = 4, 5 \dots$ represented by $\mathrm{Spin}(n)$,

$$\Delta_B \cong \begin{cases} Z(2)^2 & \text{if } n \equiv 0 \pmod 2, \\ Z(4) & \text{if } n \equiv 1 \pmod 2 \end{cases}.$$

In the first case, the outer automorphisms χ do not satisfy (7) for any a. In the second case $\chi(z) = z^3$ for any outer automorphism. If $a = 2$ then (7) is not satisfied.

Types G_2, F_4, E_7, E_8 have no outer automorphisms. The compact simply connected form L of E_6 has a center $\Delta_B \cong Z(3)$ and thus has outer nontrivial automorphisms χ with $\chi(z) = z^2$.

These remarks allow the following observation:

Example 3.14. Suppose that G is a simple connected but not simply connected compact Lie group not isomorphic to $\mathrm{SO}(3)$, $\mathrm{PSU}(3)$, E_6/Z, $\mathrm{SO}(2m)$ or a double covering of $\mathrm{SO}(2m)$. Then there is a topologically split morphism $f: A \to G$ with A locally isomorphic to G^2 such that f does not split. In particular, G is not $\mathcal{E}$-projective for the class of topologically split morphisms. ∎

We summarize:

Theorem 3.15. *Let $f: A \to B$ be a topologically split morphism of compact groups and let B be connected. Let $f_0: A_0 \to B$ denote the restriction to the identity component of A. Then*

(i) *f_0 is topologically split.*

(ii) *The induced morphism $F: A_0/A_0' \to B/B'$ is split.*

(iii) *Each isotypic $\mathfrak{s}$-component $(S_\mathfrak{s} f_0): S_\mathfrak{s}(A_0) \to S_\mathfrak{s} B$ is topologically split. The morphism $f': A' \to B'$ induced on the commutator groups is topologically split.*

The conditions (i), (ii), *and* (iii) *do not imply that f is split. In particular, a topologically split morphism between isotypic compact connected semisimple groups need not be split. Its splitting and topological splitting is characterized in* Proposition 3.13. *in terms of the projective cover.*

(iv) *If T is any connected abelian subgroup of B then the homomorphism $f^{-1}(T) \to T$ induced by f splits, as does its restriction to any maximal connected abelian subgroup of $f^{-1}(T)$.*

(v) *Every compact connected abelian group is $\mathcal{E}$-projective for the class of topologically split morphisms and therefore is a homomorphic retract of some free compact group. If G is a simple compact connected Lie group which is an $\mathcal{E}$-projective or, equivalently, which is a homomorphic retract of a free compact group, then the fundamental group $\pi_1(G)$ must be singleton or have exponent 2 or 3. For a precise listing see* Example 3.14. ∎

It is still possible that any $\mathcal{E}$-projective simple compact Lie group *has to* be simply connected. But our choice of $\kappa: L \to L$, $\kappa(g) = g^a$ will not carry us further than stated in Example 3.14 and Theorem 3.15(v).

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