VARIETIES OF TOPOLOGICAL GROUPS GENERATED BY LIE GROUPS

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1. Introduction

Varieties of topological groups have been investigated in several papers ((2) and (10)-(13)). In this note we investigate the varieties generated by classical Lie groups. In particular we show results of which the following is indicative: The variety generated by the unitary group $U(n)$ contains $U(m)$ if and only if $m \leqq n$. *En route* we introduce the notion of a variety of topological Lie algebras which provides a convenient setting in which to answer our questions.

2. Definitions and preliminary results

A non-empty class of topological groups (not necessarily Hausdorff) is said to be a *variety of topological groups* if it is closed under the operations of taking subgroups, quotients, arbitrary cartesian products and isomorphic images. The smallest variety containing a class $\mathscr{C}$ of topological groups is said to be the variety generated by $\mathscr{C}$ and is denoted by $V(\mathscr{C})$, (or $V(G)$ if $\mathscr{C} = \{G\}$).

A *Banach Lie algebra L* is a Lie algebra (over the reals) and a Banach space such that there exists a constant c such that $\|[x.y]\| \leqq c \| x \| \, \| y \|$ for each $x, y \in L$. If B_i, $i \in I$, are Banach Lie algebras and E is a subalgebra of the product $\prod_{i \in I} B_i$, then E with the topology induced from the product is said to be a (locally convex) *topological Lie algebra.* (Cf. (8) pp. 252, 253 and the fact that any locally convex Hausdorff topological vector space is a subspace of a product of Banach spaces.)

A non-empty class of topological Lie algebras is to be a *variety of topological Lie algebras* if it is closed under the formation of cartesian products, subalgebras, separated quotients and isomorphic images.

Note that the " Varieties of linear topological spaces " considered in (3) and (4) are varieties of topological Lie algebras.

If $\mathscr{C}$ is a class of topological groups then $Q\mathscr{C}$ denotes the class of all topological groups isomorphic to quotients of members of $\mathscr{C}$. Similarly we define the operators S, $\bar{S}$, $\bar{Q}$, C and D where they respectively denote subgroup, closed subgroup, separated quotient, arbitrary cartesian product and finite product.

The operators S, $\bar{S}$, $\bar{Q}$, C and D are defined similarly on classes of topological Lie algebras.

We now present two basic theorems, the first is proved in (2) and the second can be proved in a like manner.

Theorem 2.1. *If $\mathscr{C}$ is a class of topological groups and G is a Hausdorff group in $V(\mathscr{C})$, then $G \in SC\bar{Q}\bar{S}D(\mathscr{C})$.*

Theorem 2.2. *If $\mathscr{C}$ is a class of topological Lie algebras and $V(\mathscr{C})$ is the variety generated by $\mathscr{C}$, then $V(\mathscr{C}) = SC\bar{Q}\bar{S}D(\mathscr{C})$.*

The next theorem, proved in a similar manner to Theorem 4.1 of (4) has a very useful corollary.

Theorem 2.3. *Let $\mathscr{C}$ be any class of topological Lie algebras and let B be a Banach Lie algebra in $V(\mathscr{C})$. Then $B \in \bar{Q}\bar{S}D(\mathscr{C})$. In particular, this is the case for finite-dimensional topological Lie algebras.*

Corollary 2.4. *Let $\mathscr{C}$ be a class of topological Lie algebras of dimension $\leqq m$, for some integer m. Then every finite-dimensional simple topological Lie algebra L in $V(\mathscr{C})$ has dimension $\leqq m$.*

Proof. From Theorem 2.3 we see that $L \in \bar{Q}\bar{S}D(\mathscr{C})$. Using Theorem 3.1 of (7) this implies that $L \in \bar{S}D(\mathscr{C})$. Finally, noting that L is simple we have that $L \in S(\mathscr{C})$, which ensures that the dimension of $L \leqq m$.

Finally, we mention

Theorem 2.5. *Let G be a connected locally compact Hausdorff group. Then*

(i) *$V(G) \supseteq V(T)$, where T is the circle group,*

(ii) *G is compact if and only if $V(G) \not\supseteq V(R)$, where R is the group of reals,*

(iii) *G is abelian if and only if $V(G) \subseteq V(R)$,*

(iv) *If G is compact, then the variety of groups,* (14), *generated by G is either the class of all groups or the class of all abelian groups. In the latter case*

$$V(G) = V(T).$$

Proof. Suppose G is non-compact. Then, in view of § 4.13 of (9), G has R as a subgroup. Therefore $V(G) \supseteq V(R) \supset V(T)$. If G is abelian, then Theorem 5.8 of (12) yields that $V(G) = V(R)$.

Now suppose that G is compact. Then by § 4.6 of (9), G has a proper normal subgroup N such that G/N is a Lie group. This implies (§ 4.13 of (9)) G/N has T as a subgroup. Consequently, $V(G) \supseteq V(T)$. Corollary 3 of (2) states that a locally compact group in a variety generated by compact groups is compact. Thus $R \notin V(G)$.

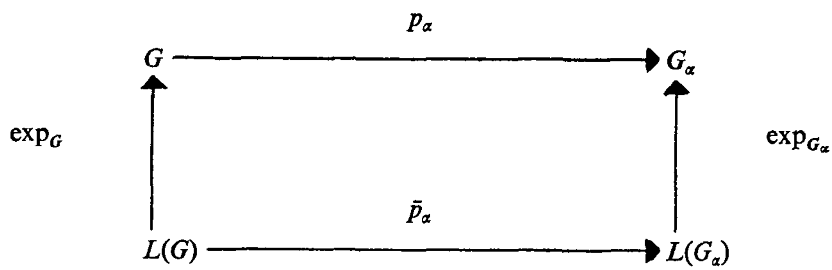
If G is compact and non-solvable, then the main result of (1) implies that the variety of groups generated by G is the class of all groups.

If G is compact and solvable, then by Theorem 29.44 of (6), G is abelian. Therefore $V(G) = V(T)$, and the variety of groups generated by G is the class of all abelian groups. This completes the proof.

3. The main results

Lemma 3.1. *Let G and $\{G_\alpha\colon\ \alpha \in I\}$ be Lie groups. If G is a subgroup of the product $\prod_{\alpha \in I} G_\alpha$, then $L(G)$ is isomorphic to a subalgebra of $\prod_{\alpha \in I} L(G_\alpha)$, where $L(G)$ and $L(G_\alpha)$ are the Lie algebras of G and G_α, $\alpha \in I$, respectively.*

Proof. Let p_α be the projection mapping of G into G_α and let $\bar{p}_\alpha$ be the induced homomorphism of $L(G)$ into $L(G_\alpha)$. Consider the diagram:



Clearly it commutes. Since there are enough maps p_α to separate points of G, there are enough maps $\bar{p}_\alpha$ to separate points of $L(G)$. Therefore the mapping of $L(G)$ into $\prod_{\alpha \in I} L(G_\alpha)$, defined to be the product of the mappings $\bar{p}_\alpha$, $\alpha \in I$, is an isomorphism of $L(G)$ on to its image.

Theorem 3.2. *Let G be a Lie group and $L(G)$ its topological Lie algebra (that is, the Lie algebra of G given the unique vector space topology it admits).*

(i) *If H is a Lie group in $V(G)$, then its topological Lie algebra $L(H)$ is in $V(L(G))$.*

(ii) *If L is any finite-dimensional topological Lie algebra in $V(L(G))$, then there is a Lie group H in $V(G)$ such that $L(H)$ is isomorphic to L.*

Proof. (i) By Theorem 2.1, $H \in SC\bar{Q}\bar{S}D(G)$. That is, $H \leqq \prod_{\alpha \in I} G_\alpha$, where each $G_\alpha \in \bar{Q}\bar{S}D(G)$. Then clearly $L(G_\alpha) \in \bar{Q}\bar{S}D(L(G))$. By Lemma 3.1, this implies that $L(H)$ is isomorphic to a subalgebra of $\prod_{\alpha \in I} L(G_\alpha)$, and hence is in $V(L(G))$.

(ii) If L is a finite-dimensional topological Lie algebra in $V(L(G))$, then, by Theorem 2.3, $L \in \bar{Q}\bar{S}D(L(G))$. This implies that there is an H in $\bar{Q}\bar{S}D(G)$ such that $L(H)$ is isomorphic to L. The proof is complete.

Corollary 3.3. *Let G be a Lie group of dimension n with a simple Lie algebra, and $\{G_\alpha\colon\ \alpha \in I\}$ be Lie groups of dimension $<n$. Then $G \notin V(\{G_\alpha\colon\ \alpha \in I\})$.*

Corollary 3.4. *Let $\mathscr{C}$ be a class of connected solvable (nilpotent) Lie groups. Then any connected Lie group in $V(\mathscr{C})$ is solvable (nilpotent).*

Proof. This follows from Theorems 3.2 and 2.3.

We now look at some examples.

Noting, (5), that the Lie algebras of the special unitary groups $SU(n)$ are simple for $n>1$, and that their dimension increases with n, Corollary 3.3 gives: $SU(m) \in V(SU(n))$ if and only if $m \leqq n$. Furthermore, noting that the unitary group $U(n)$ is a quotient group of $SU(n) \times T$ (cf. 29.48 of (6)), Theorem 2.5 shows that $V(U(n)) = V(SU(n))$ for all n. Hence $U(m) \in V(U(n))$ if and only if $m \leqq n$.

The symplectic groups $Sp(n)$ have simple Lie algebras, (5), and dimension an increasing function of n. Thus $Sp(m) \in V(Sp(n))$ if and only if $m \leqq n$.

The spin groups spin(n) have a slightly more curious behaviour. For $n \neq 2$, 4 their Lie algebras are simple and the above argument again works. We can see that spin(2) $\notin V$ (spin(1)), since spin(1) is of exponent 2 and spin(2) is not. Finally we observe that spin(4) $\in V$ (spin(3)), since spin(4) is isomorphic to spin(3) $\times$ spin(3). Thus we have spin(m) $\in V$ (spin(n)) if and only if $m \leqq n$ or $m = n+1 = 4$.

Since the orthogonal group $O(n)$ is a quotient of the product of $SO(n)$ and a suitable finite cyclic group (cf. 29.49 of (6)), we have $V(O(n)) = V(SO(n))$, for all n. Our usual argument leads to the result $SO(m) \in V(SO(n))$, if $m>n$ and $m \neq 4$.

Non-compact Lie groups such as $O(m, n)$, $U(m, n)$ and $Sp(m, n)$ can be treated similarly.

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