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Hofmann, Karl H.(D-DARM); Morris, Sidney A.(5-FLDR)
The structure of compact groups.
A primer for the student—a handbook for the expert.
de Gruyter Studies in Mathematics, 25.
Walter de Gruyter & Co., Berlin, 1998. xviii+835 pp.
DM 278.00. ISBN
3-11-015268-1

The book is devoted to the structure theory of compact topological groups, which is of special importance due to its fruitful applications, including those in harmonic analysis and representation theory. The classical structure theory of locally compact groups has deeply developed since the thirties. It includes, in particular, the structure results for compact Lie groups, the Peter-Weyl theorem and its consequences (including the properties of the irreducible characters) and two related assertions:

—(1) a compact group is a projective limit of compact Lie groups,

—(2) a compact Lie group has a faithful continuous matrix representation,

and can be found in widely dispersed monographs and textbooks including [C. Chevalley, Theory of Lie Groups. I, Princeton Univ. Press, Princeton, N. J., 1946; MR 7, 412c;

D. Montgomery and L. Zippin, Topological transformation groups, Interscience Publishers, New York, 1955; MR 17, 383b;

K. H. Hofmann and P. Mostert, Mem. Amer. Math. Soc. No. 43 (1963), 75 pp.; MR 27 #1529;

E. Hewitt and K. A. Ross, Abstract harmonic analysis. Vol. I, Academic Press, Publishers, New York, 1963; MR 28 #158;

E. Hewitt and K. A. Ross, Abstract harmonic analysis. Vol. II, Springer, New York, 1970; MR 41 #7378;

G. Hochschild, The structure of Lie groups, Holden-Day, San Francisco, 1965; MR 34 #7696;

H. Reiter, Classical harmonic analysis and locally compact groups, Clarendon Press, Oxford, 1968; MR 46 #5933;

N. Bourbaki, Elements of mathematics. Algebra, Part I, Translated from the French, Hermann, Paris, 1974; MR 50 #6689;

S. A. Morris, Pontryagin duality and the structure of locally compact abelian groups, Cambridge Univ. Press, Cambridge, 1977; MR 56 #529;

J. F. Price, Lie groups and compact groups, Cambridge Univ. Press, Cambridge, 1977; MR 56 #8743;

D. L. Armacost, The structure of locally compact abelian groups, Dekker, New York, 1981; MR 83h:22010]

and others. However, the accumulation of structure results for a compact group that are global in a sense and less related to projective limit arguments required a summarizing work. This task is performed in the book by Hofmann and Morris, which is thus "a handbook for the expert" indeed, as indicated by the authors in the expanded title. At the same time, the text is mainly self-contained, and thus the book can serve as "a primer for the student"; the prerequisites are always the minimum possible, and the four appendices,
—"Abelian groups",
"Covering spaces and groups",
"A primer of category theory", and
"Selected results in topology and topological groups",
each of which can be regarded as a good textbook, present special material that is not widely known and is helpful for reading the last chapters of the monograph. We consider now the contents of the book. The first four chapters are of preliminary nature.

Chapter 1, "Basic topics and examples", contains the main definitions, properties, and examples related to topological groups and their projective limits and "one-half" (namely, the algebraic part) of the Pontryagin duality theorem.

Chapter 2, "The basic representation theory of compact groups", introduces the Haar measure and integral and presents the main theorem on Hilbert modules for compact groups. Note that, starting from this chapter, by a compact Lie group the authors mean a compact group without small subgroups, or, equivalently, a topological group isomorphic to a compact subgroup of the multiplicative group of a Banach algebra.

In Chapter 3, "The ideas of Peter and Weyl", the first part, "The classical theorem of Peter and Weyl", includes the study of the fine structure of the representation algebra, and the second part, "The general theory of G-modules", discusses vector-valued integration and its applications, general properties of compact group actions on modules and convex cones, convolutions, and complexifications of real representations.

Chapter 4, "Characters", treats the characters of finite-dimensional representations and the structure theorems for Hilbert G-modules and for the set of elements of a G-module whose orbits span a finite-dimensional subspace; this leads to a classification of cyclic modules.

Chapters 5 and 6 present a new approach to a part of the Lie theory. The starting point of this exposition is the study of the exponential function and the logarithm in a Banach algebra, and of the local groups, the adjoint representations, and morphisms between Lie groups.

In Chapter 6, the structure theorem for a compact Lie group is proved in terms of the center, the identity component, and the commutator subgroup, together with many versions of this theorem and many special results (in particular, the case of abelian groups is investigated in detail, the closedness of the commutator subgroup of a compact Lie group is proved, the automorphism group of a compact Lie group is described, Auerbach's generation theorem is presented, and the Weyl group is investigated from different viewpoints).

Chapter 7, "Duality for abelian topological groups", and

Chapter 8, "Compact abelian groups", contain a very rich and deep exposition of the theory of locally compact abelian groups, their topology, duality, homotopy, and cohomology, and some generalizations of this theory and relationships to the set theory and logic.

The core of the book is

Chapter 9, "The structure of compact groups". For compact groups, the authors prove here five fundamental structure theorems, namely, the Levi-Maltsev theorem, the theorem on maximal pro-tori, the splitting structure theorem, the theorem on a supplementary subgroup for a maximal pro-torus, and the theorem on the structure of semisimple compact connected groups. These theorems have a rich and powerful environment. The authors also prove here important structure theorems for the exponential function and study the connectivity structure, homological algebra phenomena, and automorphism groups of compact groups (including the Iwasawa theory of the automorphism groups). The exposition here is as far from projective limit arguments as possible.

Chapter 10, "Compact group actions", presents deep results on orbits and orbit spaces, local cross sections, totally disconnected G-spaces, and compact Lie group actions on locally compact spaces; this leads to deep results on the topological decomposition of compact groups in terms of the quotient group by the connected component, the commutator subgroup of the connected component, and the quotient group of the connected component by the commutator subgroup of the connected component; the existence problem for global cross sections is studied and split morphisms are investigated.

The last two chapters,

Chapter 11 ("The structure of free compact groups") and

Chapter 12 ("Cardinal invariants of compact groups"), deal with special topics.

Each chapter and appendix has a short preface of introductory nature and a historical and theoretical postscript and is completed with a list of corresponding references. Many examples are discussed, which are mainly of general interest, and the book is endowed with many exercises, almost all of which are of the form "prove the previous assertion" (which stresses the role of the book as both a "primer for the student" and a "handbook for the expert").

Thus, the book is very informative. The text is quite understandable and well written (as are other texts by the authors). There are some misprints and mistakes, which show that the book is rather too large to be homogeneous; however, Israel Halperin (private oral communication in the early 1960s) claimed that a mathematical text must have an error on any page for a reader to be focused enough, while the book by Hofmann and Morris is very far from Halperin's "standard". Let us indicate the most "serious" mistakes.

—Lemma 11.3 claims that a certain mapping between topological groups is a homeomorphism, while in the proof of this lemma it is shown that the mapping under consideration is an isomorphism.

—In the list of symbols, the frequently used symbols $E_{\rm fin}$, $E_{\rm fix}$, and $E_{\rm eff}$ are absent (see Definition 3.1 and Definition 3.34).

—In the references, the year (1987) of reference [228] is omitted, and references [30] and [37] are related to the same book.

—In the index, "Big Peter and Weyl Theorem" differs from "Big Peter-Weyl Theorem".

—-On page 59 we can gaze at the nontrivial construction "adjoint actionadjoint action".

It should be repeated that, up to such occurrences, the exposition is excellent. The book by Hofmann and Morris is fresh and promising, and its ideas, approaches, and results advance the structure theory of more general topological groups; thus, it is guaranteed a vast audience.

Reviewed by Alexander Isaakovich Shtern